

The structure of isotropic turbulence at very high Reynolds numbers

By **ROBERT H. KRAICHNAN**

Institute of Mathematical Sciences, New York University, New York 3

(Received 24 February 1958 and in revised form 1 October 1958)

A recapitulation is first given of a recent theory of homogeneous turbulence based on the condition that the Fourier amplitudes of the velocity field be as randomly distributed as the dynamical equations permit. This theory involves the average infinitesimal-impulse-response functions of the Fourier amplitudes and employs a new kind of perturbation method which yields what are believed to be exact expansions of third- and higher-order statistical moments of the Fourier amplitudes in terms of second-order moments and these response functions.

In the present paper the theory is applied in lowest approximation (called the direct-interaction approximation) to stationary isotropic turbulence of very high Reynolds number. The characteristic wave-number $k_0 = \epsilon/v_0^3$ and Reynolds number $R_0 = v_0 k_0^{-1}/\nu$, where v_0 is the r.m.s. velocity in any given direction, ϵ is the power dissipated per unit mass, and ν is the kinematic viscosity, are introduced. For $R_0^{\frac{1}{2}} \gg 1$, it is found that the inertial and dissipation ranges extend over wave-numbers k satisfying $k_0 \ll k \ll R_0 k_0$. The time-correlation and average infinitesimal-impulse-response functions of the Fourier amplitudes in these ranges are evaluated. They are found to be asymptotically identical and given by $J_1(2v_0 k \tau)/v_0 k \tau$, where τ is the time interval.

The energy spectrum in these ranges is determined by a non-linear integral equation, involving the time-correlation and response functions, which is suitable for solution by iteration. The solution is of the form $E(k)/v_0 \nu = (k/k_d)^{-\frac{3}{2}} f(k/k_d)$, where $E(k)$ is the three-dimensional spectrum function, $k_d = R_0^{\frac{2}{3}} k_0$ is a wave-number characterizing the dissipation range, and $f(k/k_d)$ is a universal function. In the inertial range, $E(k) = f(0)(\epsilon v_0)^{\frac{1}{2}} k^{-\frac{3}{2}}$, asymptotically. The parameter $f(0)$ can be obtained by quadratures, without solving the integral equation for $E(k)$. Spectral energy transport throughout the inertial and dissipation ranges is found to proceed by a cascade process essentially local in wave-number space; the direct power delivered by all modes below k to all modes above $k' \gg k$ is of order $\epsilon(k/k')^{\frac{3}{2}}$ if k and k' both lie within the inertial range. The mean-square velocity derivatives of all orders are found to be finite. For $R_0^{\frac{1}{2}} \gg 1$ the skewness factor of the distribution of the n th-order longitudinal velocity derivative is found to have the asymptotic form $A_n R_0^{-\frac{1}{2}}$, where A_n is a universal constant.

The theory is compared with experiment and is found to be slightly better supported than the Kolmogorov theory. However, it is stressed that extreme caution must be exercised in interpreting the experimental evidence as support for either theory.

An analysis is given of the relations between the Kolmogorov theory, Heisenberg's heuristic theory, the analytical theories of Heisenberg and Chandrasekhar, the theories of Proudman & Reid and Tatsumi, and the present theory.

Contents

	PAGE
1. Introduction	498
2. Foundations of the theory	500
2.1. The maximal randomness condition	500
2.2. The weak dependence principle	503
2.3. The perturbation method	504
2.4. The direct-interaction approximation	508
3. Statistical equations for stationary isotropic turbulence	510
4. The spectral transport of energy	513
5. Response and time-correlation functions for high wave-numbers	515
5.1. The response function.	515
5.2. The time-correlation function	518
6. The inertial range	519
7. The dissipation range at huge Reynolds numbers	522
7.1. The energy-balance integral equation	522
7.2. Behaviour in the far dissipation range	523
7.3. Skewness factors for the longitudinal velocity derivatives	525
7.4. Qualitative discussion of the dynamics	526
8. Comparison with experiment	527
8.1. The effect of deviations from idealized conditions	527
8.2. Wave-number spectra in turbulent pipe flow	529
8.3. Wave-number spectra in grid turbulence	532
8.4. The response and time-correlation functions	534
9. Comparison with other theories	535
9.1. The Kolmogorov theory	535
9.2. Heisenberg's heuristic theory	538
9.3. The analytical theories of Heisenberg and Chandrasekhar	539
9.4. The theories of Proudman & Reid and Tatsumi	541
References	543

1. Introduction

An attempt has been made recently to develop an exact theory of homogeneous turbulence in which all the statistical moments of the velocity field are derived analytically from the Navier–Stokes equation (Kraichnan, 1958*a, b*; the first reference will be cited as Paper I). In the present paper an approximation to this theory is used to examine the energy dynamics and the spectral structure, with respect to both frequency and wave-number, of the inertial and dissipation ranges of very-high-Reynolds-number stationary, isotropic turbulence. The statistical equations of motion for the stationary isotropic case obtained in Paper I are not rederived here, but we shall preface our investigation with a recapitulation, and, in part, an amplification, of the foundations of the theory. This seems especially advisable since the foundations have undergone considerable evolution since the writing of Paper I; in particular, that paper contains an important misconception. A full account of the basic theory in its present state

will be given elsewhere. In discussing it here, the emphasis will be upon illuminating the principles involved, and no attempt at rigorous or complete treatment will be made.

The essential difficulties of the turbulence problem arise from the strongly dissipative character of the dynamical system and the non-linearity of the equations of motion. The first of these two characteristics effectively precludes treatment by conventional methods of statistical mechanics. The second is responsible for the fact that the Navier–Stokes equation does not yield closed differential equations for the velocity covariance, the statistical quantity of principal interest. The equations of motion for this covariance contain third-order moments of the velocity field, the equations of motion for the third-order moments contain fourth-order moments, and so forth, *ad infinitum*. A central goal of turbulence theory is the closing of this infinite chain of coupled equations into a determinate set containing only moments below some finite order.

Closure has been obtained in a number of previous treatments of homogeneous turbulence by the assumption of expressions, in terms of second-order moments, for either third-order moments (Obukhov 1941; Heisenberg 1948; von Kármán 1948; Kovásznyay 1948) or fourth-order moments (Millionshtchikov 1941; Heisenberg 1948; Proudman & Reid 1954; Chandrasekhar 1955; Tatsumi 1957). The assumptions made were justified on heuristic grounds, on the basis of experimental indications, or on grounds of mathematical simplicity.

In the present approach no *ad hoc* assumption relating higher-order to lower-order moments is made. The velocity field within a very large volume is represented by Fourier series amplitudes $u_i(\mathbf{k}, t)$, which are considered to be the fundamental dynamic variables. In addition to the moments of the many-time joint-distribution of all the Fourier amplitudes, the average impulse-response tensors of the Fourier modes are introduced. These quantities, which do not appear in the treatments cited in the last paragraph, describe the relaxation of arbitrary infinitesimal impulsive disturbances of individual vector Fourier amplitudes. The fundamental condition imposed is that the statistical dependence among the Fourier amplitudes is induced wholly by the non-linear terms in the Navier–Stokes equation and not at all by the initial conditions or by the external forces which may be acting. This condition, which we term that of maximal randomness, implies statistical homogeneity, and is more restrictive than that condition. It appears to be the simplest condition strong enough to give, in principle, a closeable statistical theory from the Navier–Stokes equation, and it seems logically indicated when the mechanism by which the turbulence is produced or maintained is not explicitly described.

Maximal randomness and certain qualitative features of the Navier–Stokes equation lead to the principle of weak dependence of the Fourier amplitudes. The essential qualitative content of this principle is that the effective dynamical coupling and statistical interdependence among any few individual Fourier amplitudes corresponding to different wave vectors is very weak when the flow volume is very large. The substantial departure from normality of the velocity distribution in x -space appears as the summed effect of very many of these very weak statistical dependence

The weak dependence principle leads to a perturbation treatment of the dynamical couplings among sets of individual Fourier amplitudes which differs importantly from conventional perturbation theory based on expansion in powers of the Reynolds number. This treatment yields what are believed to be exact infinite-series integral representations of all higher-order moments in terms of the second-order moments and the average impulse-response tensors. Integral equations fixing the second-order moments and the impulse-response tensors may then be obtained; they involve the representations for the third-order moments. If the representations are exact, these equations are also.

In order for the theory to be used, the infinite-series representations must be approximated. In the present paper, as in Paper I, the series for the third-order moments are approximated by their lowest terms. This procedure, which we term the direct-interaction approximation, has a simple dynamical significance and can be shown to lead to equations which are self-consistent in the sense that they yield rigorously realizable second-order moments. The direct-interaction approximation includes terms of all orders in an expansion in powers of the Reynolds number. It is the lowest of a sequence of self-consistent and dynamically interpretable approximations which, it is believed, should converge to yield exact and complete solutions of the statistical problem. The higher approximations involve closure of the moment-equation hierarchy at higher levels. Each successive approximation is based on new perturbation expansions, the lowest terms of which represent terms of all orders in the expansions used in the immediately preceding approximation. In each case, only these lowest terms are retained.

The reader who is not interested in the foundations of the present theory, but only in its application, may start this paper with §3. In this case the fundamental relations (3.7) and (3.11) may be regarded simply as an assumption about the form of the triple moments, analogous to those in the theories cited previously. The essential physical interpretation of this assumption, from the point of view of its consequences for the transport of energy, is then brought out in the discussion following equation (4.3).

2. Foundations of the theory

2.1. The maximal randomness condition

Let us consider a fluid confined in a cubical box of side L . Instead of requiring that the velocity vanish at the walls we shall employ the well-known artifice of cyclic boundary conditions; later, this will permit the description of rigorously statistically homogeneous flows. The velocity field $\tilde{u}_i(\mathbf{x}, t)$ may be expanded in a Fourier series

$$\tilde{u}_i(\mathbf{x}, t) = \sum_{\mathbf{k}} u_i(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (2.1)$$

where the summation is over all wave-vectors permitted by the boundary conditions. For compactness, we shall often write $\mathbf{u}(\mathbf{k})$ for $\mathbf{u}(\mathbf{k}, t)$, $\mathbf{u}'(k)$ for $\mathbf{u}(\mathbf{k}, t')$, etc. The reality of $\tilde{u}_i(\mathbf{x})$ requires $u_i(-\mathbf{k}) = u_i^*(\mathbf{k})$. In terms of the Fourier amplitudes, the Navier-Stokes equation takes the form

$$\left(\frac{\partial}{\partial t} + \nu k^2 \right) u_i(\mathbf{k}) = -\frac{i}{2} P_{ijm}(\mathbf{k}) \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} u_j(\mathbf{k}') u_m(\mathbf{k}''), \quad (2.2)$$

where ν is the kinematic viscosity and the tensor operator

$$P_{ijm}(\mathbf{k}) = k_m P_{ij}(\mathbf{k}) + k_j P_{im}(\mathbf{k}), \quad P_{ij}(\mathbf{k}) = \delta_{ij} - k^{-2} k_i k_j,$$

serves to include both Reynolds stresses and pressure forces. The latter maintain the incompressibility property

$$k_i u_i(\mathbf{k}) = 0. \quad (2.3)$$

We regard the amplitudes $\mathbf{u}(\mathbf{k})$ as the fundamental dynamic variables. Formulation of the flow problem in terms of Fourier series rather than Fourier integrals is preferred because it seems to be easier to think of dynamic variables in intuitive terms when they form a discrete set.

In all of the following we shall assume that there is no mean flow, so that $\mathbf{u}(\mathbf{0}, t) = 0$.

Let us imagine that at $t = t_0$ an arbitrary infinitesimal solenoidal perturbing force-term $\xi_i(\mathbf{k})$ is introduced on the right side of (2.2), for a single wave-vector \mathbf{k} . We may expect that the perturbation induced in $\mathbf{u}(\mathbf{k})$ will be of the form

$$\delta u_i(\mathbf{k}, t) = \int_{t_0}^t \zeta_{ij}(\mathbf{k}; t, t') \xi_j(\mathbf{k}, t') dt', \quad (2.4)$$

where $\zeta_{ij}(\mathbf{k}; t, t')$ is the infinitesimal-impulse-response tensor for mode \mathbf{k} . (We shall sometimes use the abbreviation $\zeta_{ij}(\mathbf{k})$.) If the dynamical equations were linear and time-independent, the impulse-response would be a function only of \mathbf{k} and $t - t'$. Since actually (2.2) is essentially non-linear, $\zeta_{ij}(\mathbf{k}; t, t')$ is really an implicit functional of all the Fourier amplitudes at all times between t' and t . It expresses the relaxation of an infinitesimal impulsive disturbance (imposed at t') through the joint action of viscosity and the non-linear interaction. The determination of the $\zeta_{ij}(\mathbf{k})$ is in general not less of a problem than finding the $\mathbf{u}(\mathbf{k})$ themselves. The impulse-response tensors play an essential role in the theory to be described.

Now let us take L very large compared to any characteristic length scale of the flow and consider the limit $L \rightarrow \infty$. For any flow of physical interest we may expect that in any excited wave-vector range, no matter how small, the excitation will be spread over an infinite number of individual modes in the limit. Hereafter we shall admit to consideration only such flows. This property leads to a dynamical consequence of basic importance. Any given wave-vector \mathbf{p} appears in only two terms on the right of (2.2), once as \mathbf{k}' and once as \mathbf{k}'' . In the limit $L \rightarrow \infty$ these two terms can make only an infinitesimal contribution to the summation, and hence to the time dependence of $\mathbf{u}(\mathbf{k})$ and $\zeta_{ij}(\mathbf{k})$. This suggests that the effective dynamical coupling between $\mathbf{u}(\mathbf{k})$ and $\mathbf{u}(\mathbf{p})$ ($\mathbf{k} \neq \pm \mathbf{p}$) becomes infinitely weak as $L \rightarrow \infty$. By an extension of the argument, the effective dynamical coupling among any finite set of Fourier amplitudes corresponding to distinct wave-vectors becomes infinitely weak in the limit. (We shall call a set of wave-vectors distinct when they contain no pair \mathbf{p}, \mathbf{q} such that $\mathbf{p} = \pm \mathbf{q}$.)

So far our considerations have been non-statistical. Now, in order to describe turbulence, we introduce averages $\langle \rangle$ over products of the $\mathbf{u}(\mathbf{k})$ and $\zeta_{ij}(\mathbf{k})$. The averages which appear in the present paper are to be interpreted as over an ensemble of flows, in accordance with the most usual practice in statistical

mechanics. However, for the statistically homogeneous systems which are to be treated we consider it more physical to define averages in terms of a single system (the actual physical one) by taking them over the infinitely many modes contained in appropriately chosen tiny neighbourhoods in k -space, in the limit $L \rightarrow \infty$. This single-system average corresponds closely to the experimental procedure of determining spectral quantities; we shall discuss its advantages further in a later paper. If the choice of ensemble is physically appropriate, the two kinds of average should be interchangeable for $L \rightarrow \infty$.

The fundamental restriction we shall impose on the flow is that the statistical dependence among the $\mathbf{u}(\mathbf{k})$ be induced wholly by the non-linear terms in (2.2) and not at all by initial conditions or any external forces, representable by additional terms in (2.2), which may be acting. We shall call this the maximal randomness condition. In the present paper we shall also include under this title the symmetry conditions that the real and imaginary parts of $\mathbf{u}(\mathbf{k})$ have zero means, identical auto-covariances, and are uncorrelated. Now complete statistical independence of the $\mathbf{u}(\mathbf{k})$, together with our symmetry conditions, implies statistical homogeneity.† Since the Navier–Stokes equation is invariant under translation it cannot of itself induce inhomogeneity. Hence, we may anticipate that maximal randomness restricts us to statistically homogeneous flows. The restriction actually is much more severe. Consider a flow consisting initially of an infinite array of identical vortices with centres placed at random. It is statistically homogeneous, but it is excluded because at any finite time of evolution it will display statistical dependencies among distinct wave-vectors due to the far-from-maximally random initial conditions.

The notion of maximal randomness seems to be implicit in many theoretical treatments of turbulence; few workers would call the flow just described ‘homogeneous turbulence’. We believe, however, that *explicit* appeal to this condition, or one of equivalent strength, is necessary to pose a statistical problem which is closeable even in principle. For certainly there can exist statistically homogeneous flows with a given wave-number spectrum, at some given time in their evolution, but with widely different higher statistical structures. None of these flows is excluded simply by the usual procedure of taking averages of the Navier–Stokes equation, and the future history of the various flows surely will be influenced by the differences in higher structure. Maximally random flow is a considerable idealization of laboratory homogeneous turbulence. But unless the mechanism which generates or maintains the turbulence is explicitly described by additional terms in (2.2), our condition appears to be the most reasonable way to pose a physically and mathematically determinate problem. It should be noted that maximal randomness is *not* a condition on the Reynolds number, although the correspondence to laboratory flows probably is closer at higher Reynolds numbers.

† Invariance of averages of \mathbf{x} -space quantities under translation is implied by the vanishing of all averages $\langle u_i(\mathbf{k}) u'_j(\mathbf{p}) \dots \rangle$ unless $\mathbf{k} + \mathbf{p} + \dots = 0$. Our symmetry conditions give $\langle u_i(\mathbf{k}) u'_j(\mathbf{k}) \rangle = 0$. Averages in which a given $\mathbf{u}(\mathbf{k})$ appears more than bilinearly cannot contribute to \mathbf{x} -space averages when $L \rightarrow \infty$, for the spectrally dense flows to which we have already restricted attention.

2.2. *The weak dependence principle*

We have already noted the property of weak dynamical coupling among any finite set of Fourier amplitudes in the limit $L \rightarrow \infty$. Under the maximal randomness condition, statistical interdependence among the $\mathbf{u}(\mathbf{k})$ can arise only from these weak couplings. We are led to conjecture that in this case the statistical dependencies among any finite set of $\mathbf{u}(\mathbf{k})$ become infinitely weak in the limit. A corollary is that the statistical dependencies among both the $\mathbf{u}(\mathbf{k})$ and the $\zeta_{ij}(\mathbf{k})$ for distinct wave-vectors become infinitely weak. We shall call these conjectures the weak dependence principle and state them more formally as follows.

Let the normalized Fourier amplitudes be defined as the quantities

$$\mathbf{u}(\mathbf{k}) / \langle |\mathbf{u}(\mathbf{k})|^2 \rangle^{\frac{1}{2}}.$$

Then, all moments of the many-time distribution of the response tensors and normalized Fourier amplitudes belonging to any finite set of \mathbf{k} 's tend in the limit $L \rightarrow \infty$ to values belonging to some distribution in which all quantities associated with distinct \mathbf{k} 's are statistically independent.

The following typical examples will serve to illustrate the principle. Let $\mathbf{k}, \mathbf{p}, \mathbf{q}$ be three distinct wave-vectors. Then weak dependence requires

$$\frac{\langle u_i(\mathbf{k}) u'_j(\mathbf{p}) u''_m(\mathbf{q}) \rangle}{[\langle |\mathbf{u}(\mathbf{k})|^2 \rangle \langle |\mathbf{u}'(\mathbf{p})|^2 \rangle \langle |\mathbf{u}''(\mathbf{q})|^2 \rangle]^{\frac{1}{2}}} \rightarrow 0 \quad (L \rightarrow \infty). \quad (2.5)$$

As a second example we have

$$\langle u'_i(\mathbf{k}) u_j(-\mathbf{k}) u'_m(\mathbf{p}) u_n(-\mathbf{p}) \rangle \rightarrow \langle u'_i(\mathbf{k}) u_j(-\mathbf{k}) \rangle \langle u'_m(\mathbf{p}) u_n(-\mathbf{p}) \rangle \quad (L \rightarrow \infty). \quad (2.6)$$

It is extremely important to distinguish clearly between weak dependence, which can be defined only in terms of a limiting process, and independence of the Fourier amplitudes, certain restricted consequences of which have been assumed by Millionshtchikov (1941), Heisenberg (1948), Proudman & Reid (1954), Chandrasekhar (1955), Tatsumi (1957), and other authors. It is known experimentally that the full probability distribution of the velocity field is far from normal (Batchelor 1953). In particular, the skewness and flatness factors of the difference between the velocities at two points depart from normal values. As a complement to the experimental evidence, it is clear from the Navier-Stokes equation that non-vanishing triple moments are necessary for mean energy transfer, and it has been shown from this equation that the relation of fourth-order to second-order moments is non-normal (Kraichnan 1957). By the central limit theorem, these facts are inconsistent with independence of the $\mathbf{u}(\mathbf{k})$ in the limit $L \rightarrow \infty$. However, they are wholly consistent with weak dependence. Consider the moment $M = \langle (\partial \tilde{u}_1 / \partial x_1)^3 \rangle$ which, when normalized, gives the skewness factor of the distribution of $\partial \tilde{u}_1 / \partial x_1$. Expanding, and assuming homogeneity, we have

$$M = -i \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} k_1 p_1 q_1 \langle u_1(\mathbf{k}) u_1(\mathbf{p}) u_1(\mathbf{q}) \rangle. \quad (2.7)$$

Now consider the contribution from all terms with k, p, q below some arbitrary high wave-number K . As $L \rightarrow \infty$, corresponding to a larger and larger box of

turbulence, the number of allowed wave-vectors in this range varies as L^3 and the number of such terms as L^6 . The quantities $\langle |\mathbf{u}(\mathbf{k})|^2 \rangle$ vary as L^{-3} , since

$$\frac{1}{2} \sum_{\mathbf{k}} \langle |\mathbf{u}(\mathbf{k})|^2 \rangle$$

is the energy per unit mass. Thus, M can approach a non-zero limit if the left side of (2.5) varies as $L^{-\frac{3}{2}}$, for $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$.

The case of the moment $N = \langle (\partial \tilde{u}_1 / \partial x_1)^4 \rangle$, which determines the flatness factor of the distribution of $\partial \tilde{u}_1 / \partial x_1$, is somewhat more complicated. Writing

$$N = \sum_{\mathbf{k} + \mathbf{p} + \mathbf{q} + \mathbf{m} = 0} k_1 p_1 q_1 m_1 \langle u_1(\mathbf{k}) u_1(\mathbf{p}) u_1(\mathbf{q}) u_1(\mathbf{m}) \rangle, \quad (2.8)$$

we may break the sum on the right into a part in which \mathbf{k} , \mathbf{p} , \mathbf{q} , \mathbf{m} are equal and opposite in pairs and a part in which they are not. It follows from (2.6) that as $L \rightarrow \infty$ the first part gives a contribution to N which by itself would yield the flatness factor 3, corresponding to a normal distribution of $\partial \tilde{u}_1 / \partial x_1$. In an independent distribution of the $\mathbf{u}(\mathbf{k})$ the second part would contribute zero. With weak dependence, although the terms composing this part tend individually to zero, even when normalized, their total contribution need not, just as in the case of the terms contributing to (2.7).

The examples just treated suggest that weak dependence relations of the type (2.5) may represent necessary conditions for statistical homogeneity and boundedness of x -space moments. If relations of the type (2.6) are also consequences of homogeneity,† weak dependence can be viewed as a kinematical property without reference to the Navier–Stokes equation. Our point of view here is quite different. We regard both homogeneity and weak dependence as consequences obtained by dynamical considerations from the maximal randomness condition, and, as will be discussed in § 2.3, we find weak dependence to be the key *dynamical* property which permits actual evaluation of the cross-moments of the $\mathbf{u}(\mathbf{k})$. This is also the case in certain more general applications than those to be considered here, where the maximal randomness condition is not satisfied by the initial conditions.

2.3 The perturbation method

The evaluation of the cross-moments of the $\mathbf{u}(\mathbf{k})$ with the aid of the considerations developed above involves a perturbation method with features that appear to be novel in fluid mechanics. In order to bring out the distinctive characteristics of the method with greater clarity, we first shall illustrate how one might attempt to evaluate cross-moments by conventional perturbation techniques. Let us consider the moment

$$S = \langle u_n(\mathbf{k}) u'_r(\mathbf{p}) u'_s(\mathbf{q}) \rangle \quad (\mathbf{k} + \mathbf{p} + \mathbf{q} = 0), \quad (2.9)$$

where \mathbf{k} , \mathbf{p} , \mathbf{q} are distinct wave-vectors. Moments of this type appear in the equations of motion for the covariance $\langle u'_i(-\mathbf{k}) u_n(\mathbf{k}) \rangle$. In accord with maximal randomness, let us assume that the flow is started at time t_0 with the $\mathbf{u}(\mathbf{k}, t_0)$ independently and Gaussianly distributed and with specified values of the

† There is evidence that this is the case. However, we have not found discussions of relations of this type in the literature on statistical homogeneity.

quantities $\langle u_i(\mathbf{k}, t_0) u_j(-\mathbf{k}, t_0) \rangle$ for all \mathbf{k} . Let us denote by $\bar{\mathbf{u}}(\mathbf{k})$ the amplitudes which obey $\bar{\mathbf{u}}(\mathbf{k}, t_0) = \mathbf{u}(\mathbf{k}, t_0)$ and satisfy the linearized equation of motion

$$\left(\frac{\partial}{\partial t} + \nu k^2 \right) \bar{u}_i(\mathbf{k}) = 0. \quad (2.10)$$

We may immediately solve for $\bar{\mathbf{u}}(\mathbf{k}, t)$ in terms of $\mathbf{u}(\mathbf{k}, t_0)$ and then express all the moments of the many-time distribution of the dashed quantities by using the initial conditions.

Now if an appropriate Reynolds number were sufficiently low, we could proceed by treating the non-linear terms in (2.2) as a perturbation and expanding the $\mathbf{u}(\mathbf{k})$ by iteration in terms of the $\bar{\mathbf{u}}(\mathbf{k})$. Integrating (2.2), we have

$$u_i(\mathbf{k}) = \bar{u}_i(\mathbf{k}) - \frac{i}{2} P_{ijm}(\mathbf{k}) \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \int_{t_0}^t \exp[-\nu k^2(t-t')] u'_j(\mathbf{k}') u'_m(\mathbf{k}'') dt'. \quad (2.11)$$

To obtain, on the left, the first-order approximation to $\mathbf{u}(\mathbf{k})$ we replace u on the right by \bar{u} . To continue the procedure, we successively insert on the right the expressions, for the various amplitudes, given by the previous approximation. It is clear that this results in an expansion for $\mathbf{u}(\mathbf{k})$ in ascending orders of the $\bar{\mathbf{u}}(\mathbf{k})$. For $t - t_0$ the order of some characteristic time of the turbulence this is effectively an expansion in powers of some typical Reynolds number.

The lowest-order non-vanishing contribution to S is given by

$$\bar{S}_1 = \langle \Delta u_n(\mathbf{k}) \bar{u}'_r(\mathbf{p}) \bar{u}'_s(\mathbf{q}) \rangle + \langle \bar{u}_n(\mathbf{k}) \Delta u'_r(\mathbf{p}) \bar{u}'_s(\mathbf{q}) \rangle + \langle \bar{u}_n(\mathbf{k}) \bar{u}'_r(\mathbf{p}) \Delta u'_s(\mathbf{q}) \rangle, \quad (2.12)$$

where $\Delta u_n(\mathbf{k}) + \bar{u}_n(\mathbf{k})$ is the first-order approximation to $u_n(\mathbf{k})$, etc. By (2.11) we have

$$\Delta u_n(\mathbf{k}) = \int_{t_0}^t \bar{\zeta}_{ni}(\mathbf{k}; t, t'') \bar{b}_i(\mathbf{k}, t'') dt'' + \dots, \quad (2.13)$$

with corresponding expressions for $\Delta u'_r(\mathbf{p})$, $\Delta u'_s(\mathbf{q})$, where

$$\bar{\zeta}_{ni}(\mathbf{k}; t, t'') = \delta_{ni} \exp[-\nu k^2(t-t'')]$$

and

$$\left. \begin{aligned} \bar{b}_i(\mathbf{k}) &= -i P_{ijm}(\mathbf{k}) \bar{u}_j(-\mathbf{p}) \bar{u}_m(-\mathbf{q}), \\ \bar{b}_i(\mathbf{p}) &= -i P_{ijm}(\mathbf{p}) \bar{u}_j(-\mathbf{q}) \bar{u}_m(-\mathbf{k}), \\ \bar{b}_i(\mathbf{q}) &= -i P_{ijm}(\mathbf{q}) \bar{u}_j(-\mathbf{k}) \bar{u}_m(-\mathbf{p}). \end{aligned} \right\} \quad (2.14)$$

The terms denoted by dots on the right of (2.13) involve unperturbed amplitudes with wave vectors other than $\pm \mathbf{k}$, $\pm \mathbf{p}$, $\pm \mathbf{q}$. No factors of $\frac{1}{2}$ appear in (2.14) because, to take $\mathbf{b}(\mathbf{k})$, $-\mathbf{p}$ appears on the right of (2.11) once as \mathbf{k}' and once as \mathbf{k}'' . (Note that $P_{ijm}(\mathbf{k}) = P_{imj}(\mathbf{k})$.) We have written things in a rather elaborate fashion in order to facilitate later comparisons.

The $\bar{\mathbf{u}}(\mathbf{k}, t_0)$ and hence the $\bar{\mathbf{u}}(\mathbf{k}, t)$ are statistically independent for distinct \mathbf{k} 's. Thus it is clear that only the term shown explicitly in (2.13) can contribute to the averages in (2.12). Inserting expression (2.14) in (2.13), and again noting the statistical independence property, we find for the first term on the right of (2.12) the expression

$$-i P_{ijm}(\mathbf{k}) \int_{t_0}^t \bar{\zeta}_{ni}(\mathbf{k}; t, t'') \langle \bar{u}'_r(\mathbf{p}) \bar{u}''_j(-\mathbf{p}) \rangle \langle \bar{u}'_s(\mathbf{q}) \bar{u}''_m(-\mathbf{q}) \rangle dt''. \quad (2.15)$$

This result involves only the second-order moments of the unperturbed amplitudes. Corresponding expressions may be obtained for the other two terms in (2.12).

If the appropriate Reynolds number were $\ll 1$, we should expect \bar{S}_1 to be a good approximation to S for all t . As this Reynolds number approached unity, however, we should expect that more terms in the expansions for the $\mathbf{u}(\mathbf{k})$ would have to be retained for substantial values of $t - t_0$. Finally, for the Reynolds number $\gtrsim 1$, it is not hard to see that the convergence of the expansions should be very poor if $t - t_0$ is larger than some characteristic time of the turbulence. Thus the perturbation technique just described would appear to be quite useless for cases of typical interest in turbulence theory.

The failure of conventional perturbation techniques at moderate and high Reynolds numbers of course simply reflects the fact that the non-linear terms in (2.2) are then not a small perturbation. However, we have seen that even when the non-linear terms are large they consist, in the limit $L \rightarrow \infty$, of an infinite sum of infinitesimal contributions from individual pairs of modes. This property makes it possible to develop new perturbation expansions which seem to be useful at all Reynolds numbers.

Let us again consider the moment S for the flow defined above. According to the maximal randomness condition, S must represent a phase correlation of the modes \mathbf{k} , \mathbf{p} , \mathbf{q} induced by the dynamical interaction; according to the weak dependence principle this phase correlation is infinitesimal. We may ask how the dynamical interaction produces the weak phase correlation. The most obvious way is through the *direct interaction* of $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(\mathbf{p})$, $\mathbf{u}(\mathbf{q})$, involving the terms in the equations of motion of these three amplitudes bilinear in $\mathbf{u}(\pm \mathbf{k})$, $\mathbf{u}(\pm \mathbf{p})$, $\mathbf{u}(\pm \mathbf{q})$. Denoting these terms by $\mathbf{b}(\mathbf{k})$, $\mathbf{b}(\mathbf{p})$, $\mathbf{b}(\mathbf{q})$, we have from (2.2)

$$\left. \begin{aligned} b_i(\mathbf{k}) &= -iP_{ijm}(\mathbf{k}) u_j(-\mathbf{p}) u_m(-\mathbf{q}), \\ b_i(\mathbf{p}) &= -iP_{ijm}(\mathbf{p}) u_j(-\mathbf{q}) u_m(-\mathbf{k}), \\ b_i(\mathbf{q}) &= -iP_{ijm}(\mathbf{q}) u_j(-\mathbf{k}) u_m(-\mathbf{p}). \end{aligned} \right\} \quad (2.16)$$

Let us denote by $\delta\mathbf{u}(\mathbf{k})$, $\delta\mathbf{u}(\mathbf{p})$, $\delta\mathbf{u}(\mathbf{q})$ the perturbations in the three amplitudes induced by the direct interaction. At this point very much hinges upon the meaning given the word 'induced'. If we were to follow the spirit of the conventional perturbation approach, we would take $\delta\mathbf{u}(\mathbf{k})$ to be the increment which must be added to $\bar{\mathbf{u}}(\mathbf{k})$ to yield the $\mathbf{u}(\mathbf{k})$ generated by retaining, of the non-linear parts of the equations of motion, *only* the terms $\mathbf{b}(\mathbf{k})$, $\mathbf{b}(\mathbf{q})$, $\mathbf{b}(\mathbf{p})$. Instead, we define $\delta\mathbf{u}(\mathbf{k})$, $\delta\mathbf{u}(\mathbf{p})$, $\delta\mathbf{u}(\mathbf{q})$ as the differences between the *exact* amplitudes $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(\mathbf{p})$, $\mathbf{u}(\mathbf{q})$ and the values they would have if the terms $\mathbf{b}(\mathbf{k})$, $\mathbf{b}(\mathbf{p})$, $\mathbf{b}(\mathbf{q})$ were removed from the equations of motion *and all other non-linear terms retained*. The dynamical significance of our new procedure is that instead of treating in isolation the effect of the direct interaction of modes \mathbf{k} , \mathbf{p} , \mathbf{q} we take full account of the modification of this effect due to the coupling of each of the three modes to all the rest of the system. The latter coupling is very strong at high Reynolds numbers, and we therefore expect a marked improvement over the conventional procedure.

In the limit $L \rightarrow \infty$, $\delta \mathbf{u}(\mathbf{k})$, $\delta \mathbf{u}(\mathbf{p})$, $\delta \mathbf{u}(\mathbf{q})$ represent infinitesimal perturbations according to our previous considerations. It follows that the contribution S_1 which the direct interaction makes to S in the limit is

$$S_1 = \langle \delta u_n(\mathbf{k}) u'_r(\mathbf{p}) u'_s(\mathbf{q}) \rangle + \langle u_n(\mathbf{k}) \delta u'_r(\mathbf{p}) u'_s(\mathbf{q}) \rangle + \langle u_n(\mathbf{k}) u'_r(\mathbf{p}) \delta n'_s(\mathbf{q}) \rangle. \quad (2.17)$$

Now we have

$$\delta u_n(\mathbf{k}) = \int_{t_0}^t \zeta_{ni}(\mathbf{k}; t, t'') b_i(\mathbf{k}, t'') dt''. \quad (2.18)$$

since, by (2.4), $\zeta_{ni}(\mathbf{k}; t, t'')$ is defined as the response tensor, for arbitrary infinitesimal perturbation, associated with the exact equation of motion (2.2). Thus, noting (2.16), the first term on the right of (2.17) may be written

$$-iP_{ijm}(\mathbf{k}) \int_{t_0}^t \langle \zeta_{ni}(\mathbf{k}; t, t'') u'_r(\mathbf{p}) u''_j(-\mathbf{p}) u'_s(\mathbf{q}) u''_m(-\mathbf{q}) \rangle dt''. \quad (2.19)$$

According to the weak dependence principle, the average in (2.19) is equal, in the limit, to the product of independent averages over the factors associated with $\pm \mathbf{k}$, $\pm \mathbf{p}$, $\pm \mathbf{q}$. Hence we obtain

$$-iP_{ijm}(\mathbf{k}) \int_{t_0}^t \langle \zeta_{ni}(\mathbf{k}; t, t'') \rangle \langle u'_r(\mathbf{p}) u''_j(-\mathbf{p}) \rangle \langle u'_s(\mathbf{q}) u''_m(-\mathbf{q}) \rangle dt'', \quad (2.20)$$

with corresponding results for the other two terms in (2.17). Thus we obtain an expression for S_1 which involves the exact second-order moments and average impulse-response tensors.

The direct-interaction contribution S_1 is the first term in a new, well-defined perturbation expansion for the moment S in the limit $L \rightarrow \infty$. Each higher term corresponds to the interaction of modes \mathbf{k} , \mathbf{p} , \mathbf{q} through some finite set of the modes other than \mathbf{k} , \mathbf{p} , \mathbf{q} . In each case the relevant set of bilinear interaction terms represents an infinitesimal perturbation on the *exact* equations of motion, and each resulting higher-order contribution to S is an integral expression involving only second-order moments and average response tensors.† We shall not be explicitly concerned with the higher-order terms in the present paper, and consequently we shall reserve the derivation of the full expansion for elsewhere (see Kraichnan 1958*b* for a preliminary and partial description).

The perturbation method just described can be extended to give well-defined expansions for all higher moments of the $\mathbf{u}(\mathbf{k})$ in terms of only second-order moments and the average response tensors. In the present paper, however, we shall be concerned only with the third-order moments.

The structure of (2.20) closely resembles that of (2.15), the expression for the first-order contribution according to the conventional perturbation theory. The difference is that (2.15) contains the unperturbed covariances and response tensors while (2.20) contains the exact covariances and response tensors. These latter quantities can be expanded, by the conventional procedure, in the form of

† It was erroneously concluded in Paper I (§5.2) that the higher-order contributions could vanish by symmetry. This is not the case, and equation (3.12) of Paper I, which led to the conclusion, is wrong.

infinite series in the unperturbed quantities. If this is done, and the results substituted into (2.20), it is easy to see that (2.20) contains contributions of all orders in the conventional expansion for S . It represents, in fact, a partial summation of that expansion. Consequently, we may hope that S_1 will represent a valid approximation to S at Reynolds numbers where the convergence of the conventional expansion is hopelessly poor. Each higher term in our new expansion also represents an infinite summation over terms of the old. The new expansion may be regarded formally as a regrouping and consolidation of the terms in the conventional expansion.

2.4. The direct-interaction approximation

From (2.2) we have the equation of motion

$$\frac{\partial}{\partial t} \langle u'_n(-\mathbf{k}) u_i(\mathbf{k}) \rangle = -\frac{i}{2} P_{ijm}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \langle u'_n(-\mathbf{k}) u_j(\mathbf{p}) u_m(\mathbf{q}) \rangle \quad (2.21)$$

for the covariance tensor. There is also a second equation of motion involving differentiation with respect to t' . If we substitute for the triple moments on the right the new perturbation expansions just described, we obtain an integro-differential equation which involves only the covariance tensors and the average response tensors. Further application of our new perturbation method yields an equation of motion for $\langle \zeta_{ij}(\mathbf{k}) \rangle$ which also involves an expansion in only these two kinds of quantities. Thus the method yields formally complete equations for the covariance tensors and average response tensors. We shall now outline the derivation of the direct-interaction contribution to the equation of motion for $\langle \zeta_{ij}(\mathbf{k}) \rangle$. The method to be followed differs somewhat from that employed in Paper I and has the advantage of being less formal.

Let us consider the effect of the impulsive disturbance represented by the addition of the terms $\xi_1(\mathbf{k}, t) = \alpha \delta(t - t')$, $\xi_2(\mathbf{k}, t) = \xi_3(\mathbf{k}, t) = 0$ on the right side of (2.2) for a single mode \mathbf{k} only. Here α is an infinitesimal parameter. For $t > t'$, the averaged perturbation in (2.2) has the form

$$\left(\frac{\partial}{\partial t} + \nu k^2 \right) \langle \delta u_i(\mathbf{k}) \rangle = -i P_{ijm}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \langle u_j(\mathbf{p}) \delta u_m(\mathbf{q}) \rangle, \quad (2.22)$$

where $\delta \mathbf{u}(\mathbf{k})$, $\delta \mathbf{u}(\mathbf{q})$ are the changes in $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(\mathbf{q})$ induced as a result of the initial disturbance and we retain only the terms linear in α noting the symmetry of $P_{ijm}(\mathbf{k})$ in j and m . Since the disturbance was applied originally to $\mathbf{u}(\mathbf{k})$ only, we anticipate from the property of weak coupling that each $\delta \mathbf{u}(\mathbf{q})$ is infinitesimal compared to $\delta \mathbf{u}(\mathbf{k})$ when $L \rightarrow \infty$. Now we wish to evaluate the contribution to any given term on the right of (2.22) due to direct interaction of the modes \mathbf{k} , \mathbf{p} , \mathbf{q} . This interaction produces a perturbation

$$-i P_{nrs}(\mathbf{q}) [u_r(-\mathbf{p}) \delta u_s(\mathbf{k}) + u_s(\mathbf{k}) \delta u_r(-\mathbf{p})] \quad (2.23)$$

on the right side of the equation of motion for $u_n(\mathbf{q})$. By the weak coupling property, $\delta \mathbf{u}(-\mathbf{p})$ is infinitesimal compared to $\delta \mathbf{u}(\mathbf{k})$, and we neglect it. Then, by (2.4),

$$\delta u_m(\mathbf{q}) = -i P_{nrs}(\mathbf{q}) \int_{t'}^t \zeta_{mn}(\mathbf{q}; t, t'') u_r''(-\mathbf{p}) \delta u_s''(\mathbf{k}) dt'', \quad (2.24)$$

whence

$$\begin{aligned} \langle u_j(\mathbf{p}) \delta u_m(\mathbf{q}) \rangle &= -iP_{nrs}(\mathbf{q}) \int_{t'}^t \langle \zeta_{mn}(\mathbf{q}; t, t'') u_j(\mathbf{p}) u_r''(-\mathbf{p}) \delta u_s''(\mathbf{k}) \rangle dt'' \\ &= -iP_{nrs}(\mathbf{q}) \int_{t'}^t [\langle \zeta_{mn}(\mathbf{q}; t, t'') \rangle \langle u_j(\mathbf{p}) u_r''(-\mathbf{p}) \rangle \langle \delta u_s''(\mathbf{k}) \rangle] dt'' \end{aligned} \quad (2.25)$$

by the weak dependence principle. By the definition of the response function we have $\delta u_i(\mathbf{k}) = \alpha \zeta_{i1}(\mathbf{k}; t, t')$ for the impulsive disturbance chosen. Thus we have obtained the direct-interaction contribution to the equation of motion for $\langle \zeta_{i1}(\mathbf{k}; t, t') \rangle$, and it involves only covariance tensors and average response functions.

The procedure we have just followed has a rather simple dynamical interpretation. The response tensor describes the relaxation of an initial impulsive disturbance through two effects: viscous decay and the energy exchange among modes. Viscous decay alone would yield the response tensor $\delta_{ij} \exp[-\nu k^2(t-t')]$ which appeared in the conventional perturbation theory. The transfer of initial excitation energy from mode \mathbf{k} to mode \mathbf{q} by direct interaction involves two conceptual steps. First, a perturbation is induced in mode \mathbf{q} . This is determined by the response tensor of mode \mathbf{q} and involves the amplitude of a third mode \mathbf{p} as a modulating factor because of the non-linearity of the equations of motion. Secondly, the perturbation induced in mode \mathbf{q} reacts upon mode \mathbf{k} subtracting energy out of the original excitation. The reaction again involves the amplitude of mode \mathbf{p} as a modulating factor.

The full integro-differential equations for the covariance tensors and average response tensors cannot be used without either summing or approximating the infinite series of perturbation contributions they contain. In the present paper, as in Paper I, we shall make the approximation of retaining in these equations only the direct-interaction contributions. The arguments which justify this approximation and suggest the limits of its validity can be developed properly only when the complete perturbation theory is presented, which is not the object of the present paper. We shall merely state here, without proof or amplification, some of the pertinent results. First, it can be established that the direct-interaction approximation gives self-consistent equations for the covariances and average response tensors. By this we mean that the equations yield solutions which obey the initial or other boundary conditions, fall properly to zero for infinite difference times, display rigorously positive frequency spectra and are otherwise well behaved. Furthermore, the energy conservation properties of the non-linear interaction are exactly preserved so that a self-consistent energy dynamics results. These results are strong evidence for the naturalness of the approximation.

The direct-interaction approximation appears to be a useful one at all Reynolds numbers. This is suggested by the fact that no condition on the Reynolds number is invoked in the perturbation procedure and also by the fact that the direct-interaction contributions represent contributions of all orders in the Reynolds number. We shall see in the present paper that the approximation gives plausible and self-consistent results for very high Reynolds numbers, and in a later paper we shall present semiquantitative estimates of the actual errors involved.

One might expect that better approximations to the integro-differential equations would be obtained by retaining terms of successively higher perturbation orders in the expansions for the third-order moments. This, unfortunately, is not the case because the resulting equations do not have the self-consistency properties described above, and, in fact, do not seem to have solutions obeying reasonable boundary conditions. In order to obtain self-consistent higher approximations it is necessary to close off the hierarchy of moment equations at higher levels. Each such approximation depends on new perturbation expansions the lowest terms of which represent terms of all orders in the expansions for the immediately preceding approximation. In each case only these lowest terms are retained. The approximation next above the direct-interaction approximation involves expanding fourth-order moments in terms of third-order moments, second-order moments, and certain higher response functions and thereby obtaining a complete set of equations which determine the third-order and second-order moments. The complexity of the self-consistent approximations increases rapidly with order.

In concluding discussion of the foundations of the theory, we shall mention some possible generalizations and additional applications. The theory seems capable of generalization to problems like turbulent flow in an infinite uniform channel or pipe in which there is homogeneity in only one or two directions. There is no restriction to stationarity in time, and presumably the theory could then be used to investigate not only fully developed turbulence but also the development of turbulence from the laminar state under perturbations statistically homogeneous in the axial direction of the channel or pipe.

With further modification the theory seems applicable to classes of problems which display no homogeneity at all but do exhibit statistical stationarity in time. In this case, the weak dependence principle would apply to the *frequency* components of the dynamic variables. A simple example would be a damped non-linear oscillator excited by Gaussian noise.

3. Statistical equations for stationary isotropic turbulence

The theory described in the preceding section takes its simplest form when the turbulence is isotropic in space and stationary in time. For this case each tensor involved can be expressed in terms of a single scalar, and all statistical moments depend only on difference times rather than absolute times. The remainder of the present paper will be concerned almost exclusively with the stationary isotropic case. However, we should note at the outset that this implies a considerable idealization of physical flows. For turbulence to be stationary there must be driving forces to replace energy lost by viscosity; in order to maintain isotropy the driving forces must themselves be isotropic, and this is physically unrealistic. (We may visualize such forces, perhaps, as due to a random volume distribution of stirring devices.) The physical justification for studying the stationary isotropic case (apart from its intrinsic dynamical interest) is that the structure at high wave-numbers, with which we shall be principally concerned, plausibly may be expected to differ inappreciably for stationary and freely decaying turbulence at high Reynolds numbers. We shall return to this matter after exploring the stationary theory.

In order to describe the stationary isotropic case explicitly, we replace (2.2) with the new equation of motion

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) = -ik_m P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p}, t) u_m(\mathbf{q}, t) + f_i(\mathbf{k}, t), \quad (3.1)$$

where $f_i(\mathbf{k}, t)$ is a statistically stationary and isotropic solenoidal forcing term. In accord with the condition of maximal randomness, we take the $\mathbf{f}(\mathbf{k})$ statistically independent for distinct \mathbf{k} 's.

The conditions of stationarity and isotropy require that the covariances constructed from $\mathbf{u}(\mathbf{k})$ and $\mathbf{f}(\mathbf{k})$ have the forms

$$\left. \begin{aligned} (L/2\pi)^3 \langle u_i(\mathbf{k}, t + \tau) u_j^*(\mathbf{k}, t) \rangle &= \frac{1}{2} P_{ij}(\mathbf{k}) U(k, \tau), \\ (L/2\pi)^3 \langle f_i(\mathbf{k}, t + \tau) f_j^*(\mathbf{k}, t) \rangle &= \frac{1}{2} P_{ij}(\mathbf{k}) F(k, \tau), \\ (L/2\pi)^3 \operatorname{Re} \langle f_i(\mathbf{k}, t + \tau) u_j^*(\mathbf{k}, t) \rangle &= \frac{1}{2} P_{ij}(\mathbf{k}) G(k, \tau), \end{aligned} \right\} \quad (3.2)$$

where the scalars U, F, G are all real and U, F are even functions of τ .† The normalization in (3.2) is such that in the limit $L \rightarrow \infty$ (which is required for rigorous isotropy),

$$U(k, \tau) = (2\pi)^{-3} \int \tilde{U}(x, \tau) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x, \quad (3.3)$$

where $\tilde{U}(x, \tau) = \langle \tilde{u}_i(\mathbf{x} + \mathbf{y}, t + \tau) \tilde{u}_i(\mathbf{y}, t) \rangle$, with corresponding expressions for $F(k, \tau)$ and $G(k, \tau)$.

From (3.1) we may readily obtain the equation of motion

$$\dot{U}(k, \tau) + \nu k^2 U(k, \tau) = S(k, \tau) + G(k, \tau), \quad (3.4)$$

where the dot signifies differentiation with respect to τ and

$$S(k, \tau) = (L/2\pi)^3 \operatorname{Im} \left[k_m \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \langle u_i(\mathbf{p}, t) u_m(\mathbf{q}, t) u_i^*(\mathbf{k}, t - \tau) \rangle \right]. \quad (3.5)$$

The perturbation method described in §2 may be used to expand the triple moment on the right side of (3.5). First, we note that the isotropy and stationarity of the turbulence imply isotropy and stationarity of the average of the infinitesimal-impulse-response tensor defined by (2.4). Thus we have

$$\langle \zeta_{ij}(\mathbf{k}; t, t') \rangle = P_{ij}(\mathbf{k}) g(k, t - t'). \quad (3.6)$$

The scalar $g(k, \tau)$ will be called simply the impulse-response or response function; it can be shown to be real. The factor $P_{ij}(\mathbf{k})$ in (3.6) expresses the infinite impedance of the system to non-solenoidal (compressive) forces. The direct-interaction contribution to $S(k, \tau)$ in the limit $L \rightarrow \infty$ is (see Paper I, §9.1)

$$S(k, \tau) = \pi k \int_{\Delta} \int_{\Delta} p q d p d q \int_0^{\infty} d s [a(k, p, q) g(k, s) U(p, s + \tau) U(q, s + \tau) - b(k, p, q) g(p, s) U(k, s - \tau) U(q, s)], \quad (3.7)$$

† In Paper I, the present $f_i(\mathbf{k}, t)$ was denoted by $F_i(\mathbf{k}, t)$ in order to avoid confusion with other notation. $F(k, \tau)$ has the same meaning in Paper I as it does here.

where the domain of integration extends over all wave-numbers p, q such that p, q, k can form the legs of a triangle. The geometrical factors $a(k, p, q)$ and $b(k, p, q)$ are given by

$$\left. \begin{aligned} a(k, p, q) &= \frac{1}{2}(1 - xyz - 2y^2z^2), \\ b(k, p, q) &= (p/k)(xy + z^2), \end{aligned} \right\} \quad (3.8)$$

where x, y, z are the cosines of the interior angles opposite the legs k, p, q , respectively.† They obey the trigonometric identities

$$\left. \begin{aligned} a(k, p, q) &= a(k, q, p) \geq 0, \\ k^2b(k, p, q) &= p^2b(p, k, q), \\ b(k, p, q) + b(k, q, p) &= 2a(k, p, q), \end{aligned} \right\} \quad (3.9)$$

which, as we shall see shortly, are connected with the energy conservation properties of the non-linear interaction.

Application of our perturbation technique to the evaluation of $G(k, \tau)$ yields the result‡

$$G(k, \tau) = \int_0^\infty g(k, s) F(k, s + \tau) ds. \quad (3.10)$$

Finally, as shown in Paper I, the direct-interaction approximation gives the equation of motion

$$\begin{aligned} \dot{g}(k, \tau) + \nu k^2 g(k, \tau) &= -\pi k \iint_{\Delta} pq dp dq b(k, p, q) \\ &\times \int_0^\tau g(k, \tau - s) g(p, s) U(q, s) ds \quad (\tau \geq 0), \end{aligned} \quad (3.11)$$

for the response function. For $\tau < 0$, $g(k, \tau) = 0$, since there is no response to a perturbation before it is applied. If $F(k, \tau)$ is prescribed, (3.4), (3.7), (3.10), (3.11) form a complete system determining $U(k, \tau)$ and $g(k, \tau)$, provided suitable boundary conditions are imposed. The condition of stationarity requires

$$\dot{U}(k, 0) = 0, \quad (3.12)$$

which may be formulated by (3.4) and (3.7) as an integral condition (4.2) expressing the balance of energy. The other required boundary condition is

$$g(k, +0) = 1, \quad (3.13)$$

which follows immediately from the definition of $g(k, \tau)$ and the fact that (3.1) involves only first derivatives with respect to time.

† The expression given here for $b(k, p, q)$ is related to that in Paper I by a trigonometric identity. The author is indebted to Mr Robert Wernick for the present, simplified expression. The expressions for both $a(k, p, q)$ and $b(k, p, q)$ given in Paper I should be multiplied by $\frac{1}{2}$.

‡ The restriction on the univariate distribution of $\mathbf{f}(\mathbf{k})$ made in obtaining this result in Paper I is unnecessary. Equation (3.10) can be shown to be an exact result for $L \rightarrow \infty$ provided only that the $\mathbf{f}(\mathbf{k})$ are statistically independent for distinct \mathbf{k} 's.

4. The spectral transport of energy

The covariance scalars introduced in the last section may be written

$$\left. \begin{aligned} \frac{1}{2}U(k, \tau) &= (4\pi k^2)^{-1} E(k) r(k, \tau), \\ F(k, \tau) &= (4\pi k^2)^{-1} F(k) \mu(k, \tau), \end{aligned} \right\} \quad (4.1)$$

where $E(k)$ is the kinetic energy spectrum function and $F(k)$ the forcing spectrum function, both per unit mass of fluid. The total kinetic energy per unit mass is $\int_0^\infty E(k) dk$. The quantity $r(k, \tau)$ is the time correlation function of the Fourier mode, normalized so that $r(k, 0) = 1$, with a corresponding interpretation for $\mu(k, \tau)$. If we rewrite (3.4) in terms of the new quantities, set $\tau = 0$, and note the symmetry of the integration domain with respect to p and q , we obtain the relation

$$2\nu k^2 E(k) = \frac{1}{2} \iint_{\Delta} S(k | p, q) dp dq + F(k) \int_0^\infty g(k, s) \mu(k, s) ds. \quad (4.2)$$

where

$$\begin{aligned} S(k | p, q) &= (k/pq) \{2k^2 a(k, p, q) \theta(k, p, q) E(p) E(q) \\ &\quad - [p^2 b(k, p, q) \theta(p, k, q) E(q) + q^2 b(k, q, p) \theta(q, k, p) E(p)] E(k)\} \end{aligned} \quad (4.3)$$

and

$$\theta(k, p, q) = \int_0^\infty g(k, s) r(p, s) r(q, s) ds. \quad (4.4)$$

The left side of (4.2) is the mean power dissipated by viscosity, per unit mass and per unit wave-number. The second term on the right may be interpreted as the mean power input by the driving forces, per unit mass and per unit wave-number. (Hereafter, all powers will be understood to refer to unit mass.) It should be noted that the linear dependence of this term on $F(k)$ is only apparent; the response function $g(k, s)$ is an implicit functional of F and μ determined by (3.4), (3.10) and (3.11). The first term on the right may be interpreted as a mean power input due to non-linear interaction of the velocity modes. Thus,

$$S(k | p, q) dk dp dq$$

represents the mean power delivered to the Fourier modes in the interval dk as a consequence of their triad interactions with all pairs of modes of which one member lies in each of the intervals dp and dq . $S(k | p, q)$ is symmetric in p and q , and the factor $\frac{1}{2}$ in (4.2) occurs because each pair of modes is counted twice in the integration.

It may be deduced readily from (3.1) that the interaction of any triad of Fourier modes is individually conservative; this implies

$$S(k | p, q) + S(p | q, k) + S(q | k, p) = 0, \quad (4.5)$$

which may also be obtained easily from (4.3) by using the trigonometric identities (3.9) and the fact that $\theta(k, p, q)$ is symmetric in p and q . The overall conservation property

$$\frac{1}{2} \int_0^\infty dk \iint_{\Delta} S(k | p, q) dp dq = 0 \quad (4.6)$$

follows directly from the detailed conservation relation.

It should be noted that while the concept of the elementary interaction of a *triad* of Fourier modes has a particular significance because of the detailed conservation property, the same is not true of the concept of exchange of energy between *pairs* of modes, which has been used frequently in turbulence theory. There is no way to tell how much of the energy a mode k receives from interaction with modes p and q comes from p rather than from q (Paper I, footnote 7).

The quantities which enter expression (4.3) may be given fairly direct physical interpretations. We shall consider first the factors $\theta(k, p, q)$, $\theta(p, k, q)$, $\theta(q, k, p)$ which, from their dimensions, evidently represent characteristic times for the triad interaction. We may anticipate that these factors are essentially positive. The physical significance of the response function $g(k, s)$ is that it traces the relaxation of an initial applied excitation of mode k through the joint action of viscous decay and the energy exchange, or mixing action, with the infinitely large number of only weakly statistically inter-related other modes. We may expect that it falls to zero with increasing s in a fairly smooth fashion and that it does not exhibit strongly negative regions. The correlation function $r(k, s)$ also is determined by the general mixing action. We may expect that it, too, is essentially positive and falls fairly smoothly to zero with increase of s . Thus $\theta(k, p, q)$, $\theta(p, k, q)$ and $\theta(q, k, p)$ may be expected to be the order of characteristic correlation and relaxation times of the three modes involved.

The proportionality of the terms in $S(k|p, q)$ to the θ factors may be interpreted as follows. A non-zero mean transfer of energy among the three modes requires that certain phase relations among these modes be established in a statistical sense. These relations are built up by the direct interaction of the three modes, according to the direct-interaction approximation, but are simultaneously broken down by the viscous decay and effectively random mixing with all the other modes, as expressed by the g and r functions. Loosely speaking, we may say that $\theta(k, p, q)$ is an effective time during which the triple phase correlation can build up before it is broken down.

The first of identities (3.9) indicates that $a(k, p, q)$ is never negative, and the third that $b(k, p, q)$ is typically positive. If, as anticipated, the θ factors are positive, the term in (4.3) involving a represents a *positive* flow of energy to mode k while those involving b represent a typically *negative* flow. The net flow is the resultant of these *absorption* and *emission* terms. It will be noticed that in contrast to the absorption term the emission terms are proportional to $E(k)$. This indicates that the energy exchange acts to maintain equilibrium. If the spectrum level were suddenly raised to much higher than the equilibrium value in a narrow neighbourhood of k , the emission terms would be greatly increased while the absorption term would be little affected; thus, energy would be drained from the neighbourhood and equilibrium re-established. The structure of the emission and absorption terms is such that in general we may expect the energy flow to be from strongly to weakly excited modes, in accord with general statistical mechanical principles.

Although, as was noted above, it is impossible to define unambiguously the transfer of energy between a pair of modes, it is meaningful to define a *transport power* $\Pi(k)$ as the mean power input to all modes of wave-number higher than

some value k from all modes of wave-number less than k . For this purpose we divide all the triad interactions into four classes as shown in figure 1. Classes A and B do not transfer energy across the boundary. $\Pi(k)$ is therefore the power input to all modes $k' > k$ by all interactions in class C less the power input to all modes $k' < k$ by all interactions in class D . Thus

$$\Pi(k) = \frac{1}{2} \int_k^{2k} dk' \iint_{\Delta'}^{(p, q < k)} S(k'|p, q) dp dq - \frac{1}{2} \int_0^k dk' \iint_{\Delta'}^{(p, q > k)} S(k'|p, q) dp dq. \tag{4.7}$$

The upper limit on the first integral sign is taken as $2k$ instead of ∞ because for $k' > 2k$ there are no p and q such that $p, q < k$ and k', p, q form a triangle.

It follows from the conservation properties that the interaction power-input density appearing in (4.2) is related to $\Pi(k)$ by

$$\frac{1}{2} \iint_{\Delta} S(k|p, q) dp dq = -\frac{\partial \Pi(k)}{\partial k}. \tag{4.8}$$

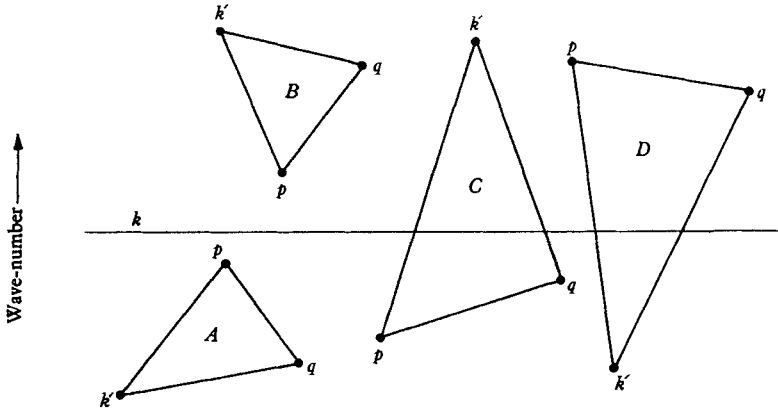


FIGURE 1. Classification of triad interactions for computing the transport power $\Pi(k)$.

5. Response and time correlation functions for high wave-numbers

5.1. The response function

The equations of motion (3.4) and (3.11), with $S(k, \tau)$ and $G(k, \tau)$ given by (3.7) and (3.10), are of a type which does not seem to have been studied, and it may be anticipated that in general their solution will present severe difficulties. Our procedure here will necessarily be mathematically non-rigorous. We shall surmise properties of the solution and then examine the consistency of our assumptions with the equations. Our primary purpose in this initial investigation is to see whether the equations seem to yield physically sensible results.

The simplest starting point is the equation of motion for $g(k, \tau)$. Using (4.1) we may rewrite (3.11) in the form

$$\begin{aligned} \dot{g}(k, \tau) + \nu k^2 g(k, \tau) \\ = -\frac{k}{2} \iint_{\Delta} dp dq \frac{p}{q} b(k, p, q) E(q) \int_0^{\tau} g(k, \tau-s) g(p, s) r(q, s) ds \quad (\tau \geq 0). \end{aligned} \tag{5.1}$$

Now let us consider a k sufficiently high that all but a very small fraction of the total energy lies very far below k . Because of the factor $E(q)$, it appears plausible to assume that the right side of (5.1) should be dominated by contributions from the principal energy containing region $q \ll k$. This assumption is supported by direct inspection of the original equation of motion (3.1). It will be noticed that the coefficient $ik_m P_{ij}(\mathbf{k})$ of the bilinear terms is independent of $|\mathbf{p}|$ and $|\mathbf{q}|$. Apart from certain angular dependence effects, the coupling of the mode \mathbf{k} to modes p or $q \ll k$ appears quite as strong intrinsically as its coupling to modes $\sim k$. Thus we might expect that the contribution of the bilinear terms to the motion should be dominated by terms involving the modes which contain most of the total excitation. Also, it is clear from the appearance of the factor k_m that, in general, the rate of change of $u_i(\mathbf{k}, t)$ ascribable to the non-linear terms tends to increase with k . Thus we may anticipate that the characteristic times of the functions $g(k, s)$ and $r(k, s)$ decrease with increasing k .

Our assumptions are related to the observations in the x -space representation that the time variation of the fine structure, viewed in a fixed co-ordinate system, should be dominated by the sweeping action of the energetic large-scale motion and that small structures should be swept past fixed points more rapidly than large ones.

Returning to (5.1), we note that for $q \ll k$ the triangle relation gives $p \approx k$; then we may approximate $g(p, s)$ by $g(k, s)$, provided it is a reasonably smooth function. Since the time characterizing $r(q, s)$ should be very large compared to that of $g(k, s)$, we shall assume that in the region where $g(k, s)$ is significantly different from zero we may replace $r(q, s)$ by $r(q, 0) = 1$. Finally, we can find from the definition (3.8) that, for $k \approx p$,

$$b(k, p, q) \approx \sin^2 \beta,$$

where β is the interior angle between k and q . The wave-number integration for q in the energy range now can be readily carried out, yielding

$$\iint_{\Delta} pq^{-1} b(k, p, q) E(q) dp dq \approx \int_0^{k_e} dq E(q) \int_{k-q}^{k+q} pq^{-1} \sin^2 \beta dp \approx \frac{4}{3} k \int_0^{k_e} E(q) dq,$$

where $k_e \ll k$ is a wave-number below which lies very nearly all the energy. Then, noting

$$\int_0^{k_e} E(q) dq \approx \int_0^{\infty} E(q) dq = \frac{3}{2} v_0^2,$$

where v_0 is the r.m.s. velocity in any direction, we obtain the equation of motion

$$\dot{g}(k, \tau) + \nu k^2 g(k, \tau) = -v_0^2 k^2 \int_0^{\tau} g(k, \tau-s) g(k, s) ds \quad (\tau \geq 0). \quad (5.2)$$

If our assumptions and approximations are well-founded, this equation should be asymptotically exact, in the direct-interaction approximation, for sufficiently high k .

The solution of the Laplace transform of (5.2) for the boundary condition (3.13) is a standard form, and we thereby find

$$g(k, \tau) = \exp(-\nu k^2 \tau) J_1(2v_0 k \tau) / (v_0 k \tau) \quad (\tau \geq 0). \quad (5.3)$$

A detailed investigation of the behaviour of the right side of (5.1), with g given by (5.3) and r estimated from the results to be derived in §5.2, indicates the consistency of our assumptions and approximations. It is found that the restriction of the integration region to $q \ll k$ can give rise to appreciable fractional error in $g(k, \tau)$ only when $\tau \gg (\nu k^2)^{-1}$. For such τ , error can arise because the integral over s may be much larger for, say, $p, q \sim \frac{1}{2}k$ than for $p \approx k$. However, $g(k, \tau)$ is very small for such τ , and the possible error is unimportant. Moreover, it decreases with increasing separation of k and the energy containing region. Our further approximations appear to be justified also.

The factor $\exp(-\nu k^2 \tau)$ in (5.3) is the response function appropriate to the decay of an applied excitation by viscosity without interaction with the other modes. The factor $J_1(2v_0 k \tau)/(v_0 k \tau)$ (plotted in figure 2) represents the relaxation

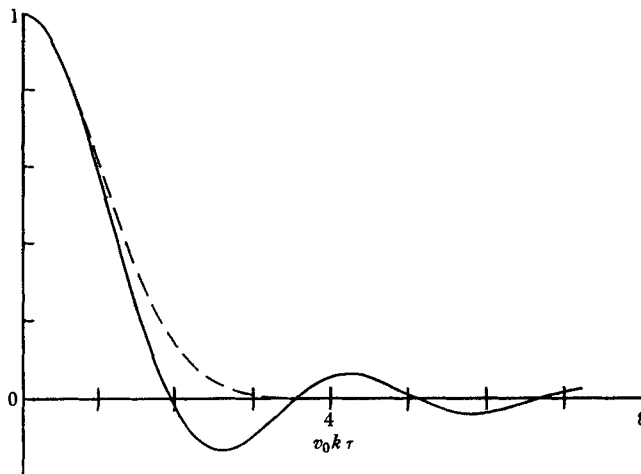


FIGURE 2. Solid curve: the asymptotic response and time-correlation function $J_1(2v_0 k \tau)/v_0 k \tau$ given by the direct-interaction approximation. Dashed curve: the function $\exp(-\frac{1}{2}v_0^2 k^2 \tau^2)$ discussed in §8.4.

associated with the energy exchange between the given mode and others with very nearly equal wave-vectors, due to the mixing action of the energy-containing region. By this process the applied excitation is mixed, or diffused, into many neighbouring (but statistically weakly dependent) modes. The characteristic time $(v_0 k)^{-1}$ associated with the mixing contribution to $g(k, \tau)$ is the same order as that which would be expected from the convection of a periodic pattern of wave-number k by a uniform velocity of magnitude v_0 . However, the action of the low wave-numbers in the present theory is very different from the convective action of a uniform field. Such a field does not produce mixing of neighbouring modes and therefore gives rise to 'coherent' time dependence rather than relaxation effects.

The oscillations in $g(k, \tau)$ which appear in figure 2 are probably to be ascribed to the direct-interaction approximation rather than the physical phenomenon. We shall discuss this in relation to experimental evidence in §8.4.

5.2. The time-correlation function

In actual stationary turbulent flows at high Reynolds numbers, the supply of energy to the turbulence seems to be concentrated principally in the energy-containing range. This suggests that in our present investigation we take $\mathbf{f}(\mathbf{k})$ to be negligible outside this range. Then for \mathbf{k} well above the energy-containing range we may omit the term $G(k, \tau)$ in (3.4). In this case (3.4), (3.7) and (4.1) lead to the equation of motion

$$\dot{r}(k, \tau) + \nu k^2 r(k, \tau) = \frac{k}{2} \iint_{\Delta} \left[k^2 a(k, p, q) \frac{E(p) E(q)}{E(k)} \int_{\tau}^{\infty} g(k, s - \tau) r(p, s) r(q, s) ds - p^2 b(k, p, q) E(q) \int_0^{\infty} g(p, s) r(k, \tau - s) r(q, s) ds \right] \frac{dp dq}{pq}. \quad (5.4)$$

(In obtaining this form a slight change of time variable has been made, and it has been noted that r is an even function of time.)

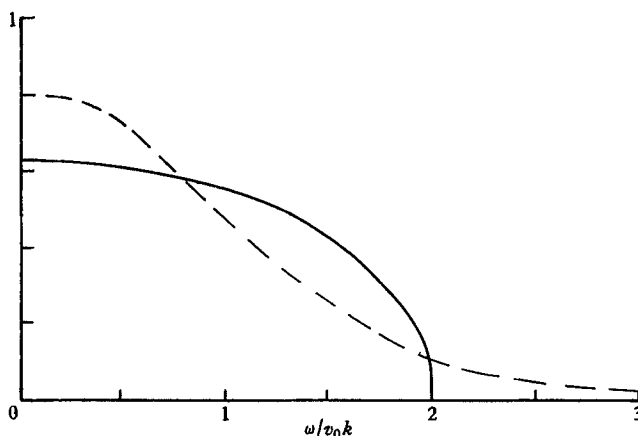


FIGURE 3. Solid curve: the (non-dimensionalized) asymptotic frequency spectrum function $(2/\pi) [1 - \omega/2v_0 k]^{\frac{1}{2}}$ given by the direct-interaction approximation. Dashed curve: the function $(2/\pi)^{\frac{1}{2}} \exp(-\frac{1}{2}\omega^2/v_0^2 k^2)$ discussed in §8.4.

Now let us assume that the Reynolds number is high enough for there to be a range of k large compared to energy-containing wave-numbers but small compared to v_0/ν , so that $\nu k^2 \ll v_0 k$. For such k we may anticipate that the behaviour of $r(k, \tau)$ should be dominated by the energy range mixing action discussed previously, and consequently be characterized by a time the order of $(v_0 k)^{-1}$. If this be so, the term $\nu k^2 r(k, \tau)$ on the left of (5.4) should be negligible compared to $\dot{r}(k, \tau)$, except in the immediate vicinity of the origin. Now, following arguments similar to those used in obtaining (5.2), let us retain on the right of (5.4) only those contributions, involving $b(k, p, q)$, such that $E(q)$ refers to the energy-containing range and only those, involving $a(k, p, q)$, such that $E(p)$ or $E(q)$ refers to this range. Making the further approximations used in obtaining (5.2), taking either $E(q)/E(k)$ or $E(p)/E(k) \approx 1$, as appropriate, using the fact that

$$2a(k, p, q) \approx \sin^2 \beta \quad (q \ll k)$$

and writing $\int_0^{\infty} ds = \int_0^{\tau} ds' + \int_{\tau}^{\infty} ds$, where $s' = \tau - s$,

we obtain the asymptotic equation

$$\dot{r}(k, \tau) = -v_0^2 k^2 \left\{ \int_0^\tau g(k, \tau - s') r(k, s') ds' + \int_\tau^\infty [r(k, \tau - s) g(k, s) - g(k, s - \tau) r(k, s)] ds \right\}. \quad (5.5)$$

The solution satisfying (3.12) is $r(k, \tau) = g(k, |\tau|)$, since for this choice the second integral on the right of (5.5) vanishes and the equation reduces to the asymptotic form taken by (5.2) for the case $\nu k^2 \ll v_0 k$. Thus we have the result

$$r(k, \tau) = J_1(2v_0 k \tau) / (v_0 k \tau). \quad (5.6)$$

A consistency check similar to that described after (5.3) suggests that this solution is indeed asymptotically valid if k satisfies the double inequality we have postulated. The frequency spectrum corresponding to our solution is

$$\begin{aligned} r(k, \omega) &= \frac{2}{\pi} \frac{1}{v_0 k} \left[1 - \left(\frac{\omega}{2v_0 k} \right)^2 \right]^{\frac{1}{2}} & (\omega \leq 2v_0 k), \\ &= 0 & (\omega > 2v_0 k). \end{aligned} \quad (5.7)$$

This function is shown, non-dimensionalized, in figure 3.

The spectrum of a correlation function must be positive everywhere, and the fact that (5.7) exhibits this property is some evidence of the consistency of our theory. The existence of all derivatives of $r(k, \omega)$ at the origin indicates that all the moments $\int_0^\infty r(k, \tau) \tau^n d\tau$ exist. The sharp cut-off in the spectrum at $\omega = 2v_0 k$ is associated with the oscillations in $r(k, \tau)$. Almost certainly, it is a consequence of the direct-interaction approximation rather than a physical fact. We shall discuss this further in §8.4.

6. The inertial range

Let us assume the existence of an extended range of k which satisfies all the inequalities invoked in §5 and over which the integral of the left side of (4.2) is negligible compared to the total dissipation at higher wave-numbers. We shall call this the inertial range. If, as before, we take the driving forces to be negligible, the energy-balance equation (4.2) reduces in this range to

$$\frac{1}{2} \iint_{\Delta} S(k|p, q) dp dq = 0. \quad (6.1)$$

It follows according to the analysis of §5 that the contributions to this equation for which p or q is in the energy-containing range are given asymptotically by $2E(k)$ times the right side of (5.5) for $\tau = 0$. The s integrals in this expression vanish identically (independently of the form of $r(k, s)$ and $g(k, s)$), which suggests that modes of high wave-number do not receive energy from direct interaction with the energy-containing range. This is in accord with the widely accepted intuitive notion that the transfer of energy up the spectrum proceeds by an essentially local cascade process. We shall assume that this is so, determine the spectrum in the inertial range, and then check the consistency of our assumption

by obtaining an expression for the energy transport due to interaction among distant wave-numbers.

The existence of an effectively local cascade process, together with the conservative character of the interaction, implies that in the inertial range, where dissipation is negligible, the transport power must satisfy

$$\Pi(k) = \epsilon, \quad (6.2)$$

where the parameter ϵ is the mean power eventually dissipated at very high wave-numbers. In accord with the concept of a local cascade process we shall assume that contributions to $\Pi(k)$, as given by (4.7) and (4.3), may be neglected if p , q , or k' is very large or small compared to k . For all the remaining contributions, the relevant response functions and correlation functions are of the form $J_1(2v_0 k\tau)/(v_0 k\tau)$. The θ factors defined by (4.4) then are symmetric in their arguments and have the dimensional form

$$[\theta(k, p, q)] = [v_0]^{-1} [k]^{-1}.$$

Consequently, $\Pi(k)$ has the dimensional form

$$[\Pi(k)] = [v_0]^{-1} [k]^3 [E(k)]^2.$$

It seems rather clear from this that if $\Pi(k)$ is to be determined wholly by contributions with k' , p and q all in some essentially local neighbourhood of k and is to have the value ϵ , which is independent of k , then one must have

$$E(k) = f(0) (\epsilon v_0)^{\frac{1}{2}} k^{-\frac{3}{2}}, \quad (6.3)$$

where $f(0)$ is a numerical constant. (We choose this symbol for later convenience.)

The argument just given only suggests the necessary form of $E(k)$, and we must check to see that (6.3) actually satisfies (6.1) and leads to a local cascade process. If (6.3) is substituted in (4.3) and one takes $g(k, s) = g(v_0 ks)$, $r(k, s) = r(v_0 ks)$ (without specifying the particular functional forms of r and g), it is not difficult to verify by formal manipulations, using the identities (3.9), that (6.1) is satisfied. This actually is rather clearly implied by the conservative nature of the theory and the dimensional considerations above. The formal property has meaning, however, only if the integrals involved converge properly, which corresponds physically to the presumed localness of cascade. We may verify the latter property by considering the total power input to all modes of wave-number $k'' > k'$ from direct interactions with all mode pairs p, q such that p or $q < k \ll k'$, where k and k' are fixed wave-numbers and all wave-numbers concerned are within the inertial range. Using $r(k, s) = g(k, |s|) = J_1(2v_0 ks)/(v_0 ks)$ and computing this power as

$$\Pi(k' | k) = \frac{1}{2} \int_{k'}^{\infty} dk'' \iint_{\Delta^*}^{(p \text{ or } q < k)} S(k'' | p, q) dp dq, \quad (6.4)$$

we find, after considerable algebra, the asymptotic result

$$\Pi(k' | k) = (\text{numerical factor}) [f(0)]^2 \epsilon (k/k')^{\frac{3}{2}} \quad (k \ll k'). \quad (6.5)$$

The triad interactions involved in the integration are shown in figure 4. $\Pi(k' | k)$ goes to zero with k/k' , so that the energy transport is asymptotically local, as

originally assumed. However, the dependence on k/k' is not particularly strong, and thus the cascade is rather diffuse.

It remains to determine the constant $f(0)$. This may be done by substituting (6.3) into (4.7), taking the form $J_1(2v_0 ks)/(v_0 ks)$ for the response and correlation functions, evaluating the definite multiple integrals, and noting (6.2). After this is done, (5.6) and (6.3) provide a complete asymptotic solution for the inertial range.

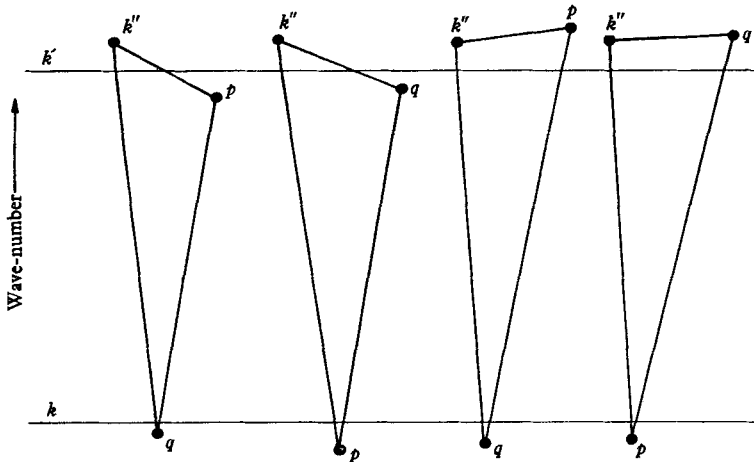


FIGURE 4. Triad interactions contributing to the distant-transport power $\Pi(k'|k)$.

If the inertial range extends through and well above the wave-number k , the total energy above k according to (6.3) is

$$\int_k^\infty E(p) dp \approx 2f(0) (\epsilon v_0)^{\frac{1}{2}} k^{-\frac{1}{2}}. \quad (6.6)$$

Now let us consider the extrapolation of this expression down to low k . Assuming that $f(0)$ is the order of unity, it is clear that (6.6) gives an energy the order of v_0^2 (per unit mass) when k is the order of

$$k_0 = \epsilon/v_0^3. \quad (6.7)$$

Such an extrapolation violates our conditions for the inertial range, of course, but nevertheless the result suggests strongly that k_0 is actually the order of a wave-number characteristic of the energy-containing range. It is also known empirically that this is the case (Batchelor 1953, p. 103). We therefore define a Reynolds number nominally characteristic of the energy-containing range by

$$R_0 = v_0 k_0^{-1}/\nu = v_0^4/\epsilon\nu. \quad (6.8)$$

If the inertial range extends through and well below a wave-number k , the total power dissipated below k is

$$\int_0^k 2\nu p^2 E(p) dp \approx \frac{4}{3} f(0) \nu (\epsilon v_0)^{\frac{1}{2}} k^{\frac{3}{2}}. \quad (6.9)$$

This expression becomes the order of ϵ for k the order of

$$k_d = \epsilon^{\frac{1}{2}} v_0^{-\frac{1}{2}} \nu^{-\frac{2}{3}} = R_0^{\frac{2}{3}} k_0. \quad (6.10)$$

It is clear, then, that necessary conditions for the validity of our asymptotic inertial range spectrum are

$$k_0 \ll k \ll R_0^{\frac{2}{3}} k_0 \quad (6.11)$$

for an extended range of k . This requires in turn that R_0 satisfy

$$R_0^{\frac{2}{3}} \gg 1. \quad (6.12)$$

Since $v_0/\nu = R_0 k_0$, it is clear that the condition $\nu k^2 \ll v_0 k$, invoked in the derivation of (5.6), is contained in (6.11).

A Reynolds number R_λ may be defined by $R_\lambda = v_0 k_\lambda^{-1}/\nu$, where $k_\lambda^2 = \epsilon/15v_0^2\nu$ (Batchelor 1953, p. 51). Thus,

$$\left. \begin{aligned} R_\lambda &= (15R_0)^{\frac{1}{2}}, \\ k_\lambda &= R_\lambda k_0/15. \end{aligned} \right\} \quad (6.13)$$

The condition (6.12) is equivalent to

$$R_\lambda^{\frac{2}{3}} \gg 1. \quad (6.14)$$

7. The dissipation range at huge Reynolds numbers

7.1. The energy-balance integral equation

The discussion at the end of §6 indicated that a major part of the dissipation takes place at wave-numbers the order of $k_d = R_0^{\frac{2}{3}} k_0$. This suggests that for $R_0^{\frac{2}{3}} \gg 1$, where the inequality is strong enough, all but a negligible fraction of the total dissipation occurs at wave-numbers $k \ll R_0 k_0 = v_0/\nu$. We shall assume that this is so and verify later that the resulting equations do yield a dissipation which falls off rapidly for $k \gg k_d$. Then for k in the dissipation as well as the inertial range, $r(k, s) \approx g(k, |s|)$ is given by (5.6). Let us anticipate, on the basis of the local cascade process already found for the inertial range, that energy transport in both inertial and dissipation ranges is negligibly dependent on direct contributions from triad interactions with modes outside both these ranges. Then the energy balance equation (4.2) takes the asymptotic form

$$2\nu k^2 E(k) = k \iint_{\Delta} [k^2 a(k, p, q) E(p) - p^2 b(k, p, q) E(k)] E(q) \theta_I(k, p, q) \frac{dpdq}{pq}, \quad (7.1)$$

$$\text{where} \quad \theta_I(k, p, q) = \int_0^\infty \frac{J_1(2v_0 ks)}{v_0 ks} \frac{J_1(2v_0 ps)}{v_0 ps} \frac{J_1(2v_0 qs)}{v_0 qs} ds. \quad (7.2)$$

As before, we have taken $F(k)$ to be negligible for $k \gg k_0$. It is clear physically that the solution of (7.1) cannot be unique, since we have not specified the power input or dissipation rate. (In effect, we have pushed the source of energy supply off to zero wave-number.) To specify a presumably unique solution we may require

$$\int_0^\infty 2\nu k^2 E(k) dk = \epsilon. \quad (7.3)$$

$$\text{If we write} \quad E(k) = (\epsilon v_0)^{\frac{1}{2}} k^{-\frac{5}{2}} f(k/k_d), \quad (7.4a)$$

which may be expressed in the dimensionless form

$$E(k)/v_0\nu = (k/k_d)^{-\frac{5}{2}} f(k/k_d), \quad (7.4b)$$

we readily find that (7.1) reduces to the universal equation

$$2u^{\frac{1}{2}}f(u) = \iint_{\Delta} \left[u^2 a(u, v, w) \frac{f(v)}{v^{\frac{1}{2}}} - v^2 b(u, v, w) \frac{f(u)}{u^{\frac{1}{2}}} \right] \frac{f(w)}{w^{\frac{1}{2}}} K\left(\frac{v}{u}, \frac{w}{u}\right) \frac{dv dw}{vw}, \quad (7.5)$$

where
$$K(m, n) = \int_0^{\infty} \frac{J_1(2s)}{s} \frac{J_1(2ms)}{ms} \frac{J_1(2ns)}{ns} ds. \quad (7.6)$$

Equation (7.3) takes the form

$$\int_0^{\infty} u^{\frac{1}{2}} f(u) du = \frac{1}{2}. \quad (7.7)$$

Instead of imposing (7.7), we could require that $f(0)$ have the value determined in §6 from (6.2). The two conditions are equivalent because the non-linear interaction is conservative; the energy transported through the inertial range is equal to the total viscous dissipation.

The singularities in the integrand of (7.5) at $v = 0$ and $w = 0$ cancel, as may be verified, with some labour, from (3.8) and (3.9). This, again, expresses the localness of the cascade process, according to which the total contribution to the integrand from the infinitesimal neighbourhood of these points is infinitesimal. We anticipate that $f(u)$ is a well-behaved function in the entire range $0 \leq u \leq \infty$.

It was pointed out in §4 that the emission term in the energy balance equation is proportional to $E(k)$, so that if the spectrum level in the immediate neighbourhood of k were suddenly raised above its equilibrium value the result would be a decrease of the net non-linear input to the neighbourhood and a return to equilibrium. This behaviour suggests that it should be possible to solve (7.5) by a corresponding correction or iteration procedure, in which one obtains an improved trial function as an appropriate linear combination of an initial trial function $2u^{\frac{1}{2}}f(u)$ and the function generated as the right side of (7.5) upon substituting the initial function. It may be expected, however, that considerable practical difficulty will arise because of the singularities in the integrand.

It is of interest at this point to discuss further the relative roles played in (5.4) by local and by energy-containing modes. We have noted that for $k \ll R_0 k_0$ the behaviour of $r(k, \tau)$ and $g(k, \tau)$ as functions of τ is determined by interactions involving the energy-containing region. However, although contributions involving the energy-containing region dominate the right side of (5.4) away from the origin, they cancel out in the immediate neighbourhood of $\tau = 0$, as indicated by (5.5). In this neighbourhood, $\dot{r}(k, \tau)$ goes to zero and the term $\nu k^2 r(k, \tau)$, which here dominates the left side, is balanced by contributions on the right nearly entirely from local triads. At $\tau = 0$, this corresponds to the localness of energy transport. As the inequalities $k_0 \ll k \ll R_0 k_0$ become stronger, the range of τ about the origin in which wholly local contributions dominate becomes a smaller fraction of $(v_0 k)^{-1}$.

7.2 Behaviour in the far dissipation range

Let us suppose that R_0 is sufficiently large that there are values of k , very large compared to k_d , for which (7.1) is still valid. There appear to be two general possibilities for the energy supply to such wave-numbers. Either they are

powered by an essentially local cascade process, or they draw power principally from wave-numbers of the order of k_d , in the region where the spectrum begins to fall below the inertial range form. We might perhaps anticipate that the first possibility is the actual fact. In the inertial range we have seen that the energy transport is essentially local. If now we imagine a part of the spectrum to be depressed by local viscous drain, it would appear that the net input from much lower wave-numbers should be affected relatively little, since to start with the local region was already very far from equipartition with these wave-numbers; on the other hand, the local decrease in the level of excitation with respect to that of closer wave-numbers should act in the direction of increasing relatively more the net input from these latter wave-numbers. These considerations actually must be considerably elaborated because of the non-linearity of the system, but it is not difficult to verify that the transport does remain essentially local in the dissipation range and to find the qualitative shape of the spectrum for $k \gg k_d$.

First, it is easy to see that no law of the form $E(k) \propto k^n$ is possible for $k \gg k_d$. Consider, to start, the assumption that the transport is essentially local so that the right side of (7.1) is dominated by contributions for which q and p are the order of k . Then in the contributing regions $\theta_I(k, p, q)$ is of order $(v_0 k)^{-1}$. Since a and b are dimensionless, one finds, counting up the wave-number factors, that (7.1) corresponds to the order-of-magnitude relation

$$2\nu k^2 E(k) \sim v_0^{-1} k^2 [E(k)]^2. \quad (7.8)$$

As k increases indefinitely this cannot be satisfied by any power law except the absurd choice $n = 0$. On the other hand, if we assume that most of the power is supplied by interactions involving modes of wave-number $\sim k_d$, then in place of (7.8) we have

$$2\nu k^2 E(k) \sim v_0^{-1} k_d k E(k_d) E(k), \quad (7.9)$$

where leading terms have been retained on the right. The expression $k_d k$ appears instead of k^2 because one of the pair p, q is of order k_d and the integration ranges of both p and q are restricted to intervals of width $\sim k_d$. Equation (7.9) cannot be satisfied by any power law at all. Thus, both (7.8) and (7.9) are inconsistent with the initial assumption of a power law, under which they were derived.

The impossibility of a power law arises from the fact that E appears linearly on the left of (7.1) and bilinearly on the right, while the equation is homogeneous in the wave-numbers. This suggests that $E(k)$ contains a factor e^{-ck/k_d} (where c is a numerical factor), since that form gives the product law

$$e^{-cp/k_d} e^{-cq/k_d} = e^{-c(p+q)/k_d}.$$

An investigation of (7.1) for $k \gg k_d$ shows that it is, in fact, satisfied asymptotically by a function of the form

$$E(k) = \text{const} (k/k_d)^2 e^{-ck/k_d} \quad (k \gg k_d). \quad (7.10)$$

The dominant contributions to the power input come from wave-number pairs p, q forming very flat triangles with k , as would be anticipated from the effect of the exponential factors in depressing the value of $E(p)E(q)$ for pairs not having this property. The integration thus is effectively confined to an area of the p, q plane

very much smaller than k^2 , and in this area $E(p)E(q) \gg [E(k)]^2$. The factor $(k/k_a)^2$, which might seem surprising in view of the homogeneity of (7.1) in wave-numbers, arises because of the restriction of integration domain; its form also is affected by the behaviour of $a(k, p, q)$, which vanishes for an exactly flat triangle. As we partially anticipated on general grounds, the energy transport is local; this appears as a consequence of the fact that as k_a/k goes to zero so does the fraction of p, q pairs satisfying $p + q \approx k$ which also satisfy p or $q \sim k_a$.

The form of (7.10) confirms the assumption that the dissipation is effectively confined to wave-numbers the order of k_a . It suggests also that each decade range (to make a somewhat arbitrary division) in the region $k \gg k_a$ dissipates most of the energy supplied to it, passing on an ever-decreasing fraction to the next decade.† Corresponding to this fact, the emission terms in (7.1) are very small compared to the absorption term for $k \gg k_a$.

An important characteristic of (7.10) is that it suggests the existence of the moments $\int_0^\infty k^n E(k) dk$ for arbitrarily high n . In the x -space representation, this implies the existence of mean square velocity derivatives of all orders—a property which seems required by physical intuition. To establish this conclusion more fully it is necessary to investigate the behaviour of $E(k)$ for extremely high wave-numbers $k \gg R_0 k_0$. Although we have not determined the form of $r(k, \tau)$ in this range, the energy balance equation is, nevertheless, tractable because the function $\theta(k, p, q)$ appearing in the dominant absorption term is determined asymptotically by $g(k, s) \approx \exp(-\nu k^2 s)$. An analysis of the type described above shows that

$$E(k) \propto k^3 \exp[-(\text{const})k] \quad (k \gg R_0 k_0). \quad (7.11)$$

This result appears to be valid even for $R_0^{\frac{1}{2}} \sim 1$. The proportionality constants, however, are not independent of R_0 .

7.3. Skewness factors for the longitudinal velocity derivatives

For $R_0^{\frac{1}{2}} \gg 1$, the skewness factor

$$S_1 = \langle (\partial \tilde{u}_1 / \partial x_1)^3 \rangle / \langle (\partial \tilde{u}_1 / \partial x_1)^2 \rangle^{\frac{3}{2}} \quad (7.12)$$

is given in terms of spectrum moments by the asymptotically valid relation (Batchelor 1953, p. 168)

$$S_1 = -\frac{3}{7}(30)^{\frac{1}{2}} \nu \left[\int_0^\infty k^2 E(k) dk \right]^{-\frac{3}{2}} \int_0^\infty k^4 E(k) dk. \quad (7.13)$$

By (7.4) and (7.7), we find

$$S_1 = -\frac{12}{7}(15)^{\frac{1}{2}} R_0^{-\frac{1}{2}} \int_0^\infty u^{\frac{5}{2}} f(u) du. \quad (7.14)$$

We can also evaluate S_1 by expressing it in terms of k space triple moments and expressing these moments in terms of $U(k, \tau)$ and $g(k, \tau)$ by the method employed in Paper I to obtain (3.7). This same procedure can be applied, in general, to the determination of the skewness factor S_n of the distribution of $\partial^n \tilde{u}_1 / \partial x_1^n$. It is not difficult to show that for $R_0^{\frac{1}{2}} \gg 1$ the result depends asymptotically only on the

† We must have an *extremely* high R_0 for this to be other than a purely academic point.

dissipation range structure and that upon substituting (5.3), (5.6) and (7.4) in the resulting expressions we obtain the asymptotic result

$$S_n = A_n R_0^{-\frac{1}{2}}, \quad (7.15)$$

where the A_n are universal constants.

The R_0 dependence indicated by (7.14) and (7.15) is quite weak, and it is impossible to estimate, without further analysis, how strong the inequality $R_0^{\frac{1}{2}} \gg 1$ has to be for the asymptotic behaviour not to be masked by other effects.

7.4. Qualitative discussion of the dynamics

An interesting paradox arises when we attempt to interpret the dynamics of huge Reynolds number turbulence, as given by the present theory, in terms of familiar impedance concepts. Equation (6.7) implies that if the driving forces were removed the characteristic decay time for the energy-containing range of the turbulence, due to the power output to higher wave-numbers, would be of the order of $(\nu_0 k_0)^{-1}$; that is, the energy range is approximately critically damped by a resistive dynamical coupling to higher wave-numbers.

Now let us go up the wave-number spectrum considering each decade, say, a subsystem. (We imagine a sufficiently huge R_0 that there are many decades in each spectrum range.) As we go through the inertial range, the characteristic time of each subsystem is of order $(\nu_0 k)^{-1}$, the energy in each subsystem is of order $(\epsilon \nu_0)^{\frac{1}{2}} k^{-\frac{1}{2}}$, and the power output to the next higher subsystem is of order ϵ . The 'damping factor' of the subsystem, the fraction of energy passed on in a characteristic 'period', is of order

$$\epsilon(\nu_0 k)^{-1} / (\epsilon \nu_0)^{\frac{1}{2}} k^{-\frac{1}{2}} = (k_0/k)^{\frac{1}{2}}.$$

Thus, the damping due to dynamical coupling goes steadily down as we rise through the inertial range.

When we get into the dissipation range $k \gtrsim k_d$, a similar argument shows that the damping due to coupling with the next higher subsystem continues to go down, and with great rapidity since the coupling to higher modes becomes ineffective and transfers a rapidly decreasing fraction of the energy in a subsystem. In this range, as in the inertial range, the damping factor associated with direct viscous dissipation increases linearly with k (as $\nu k^2/\nu_0 k$); however, it does not become critical until $k \sim R_0 k_0$, which is far above the range of significant dissipation for huge Reynolds numbers. Thus we have the situation that the energy-containing range, where viscosity does not act, is approximately critically damped, while the range in which viscosity disposes of the energy is very lightly damped. As we go up above $R_0 k_0$ the damping due to dynamical coupling continues to decrease rapidly (it has long since become extremely small), while the direct viscous damping becomes, finally, very large and dominates the mechanics of the subsystems.

The behaviour described seems strange, but there does not appear to be anything either physically or mathematically inconsistent about it. It arises from the non-linearity and complication of the system, and the consequent artificiality of dividing it into subsystems. The characteristic frequencies of the subsystems are

not really internal parameters, but are controlled by the energy-range mixing, as discussed before. We must also keep in mind that each subsystem serves as a kind of frequency multiplier for the energy supplied to it.

The asymptotic identity of time correlation and response functions for the inertial and dissipation ranges indicated by the present theory is of considerable fundamental interest. It has been shown for many types of conservative systems that time-correlation and averaged response functions are identical in an equilibrium (detailed-balance) ensemble. In particular, this is true for a dissipationless system closely related to turbulence (Paper I, appendix). In the present case we have found this same behaviour for modes which are very far from a state of detailed balance and which dissipate most of the energy lost by the system. It seems likely that this is significantly connected with the fact that for these modes the effective damping due to both dynamical coupling and direct dissipation is very small.

8. Comparison with experiment

8.1. *The effect of deviations from idealized conditions*

Our analysis has been based on maximal randomness, exact isotropy, and stationarity, and it has been limited to the direct-interaction approximation. Furthermore, we have indicated solutions of the resulting equations only for extremely high Reynolds numbers. In order to make a very meaningful comparison with experiment, it is necessary to have some idea of the theoretical corrections implied by the experimental deviations from the ideal conditions listed. It also is desirable to have theoretical estimates of the errors associated with the direct-interaction approximation. The latter estimates will be reserved for a future paper. The principal question is whether the qualitative results of §§6 and 7 and the spectrum form (7.4*b*) are consequences also of the exact asymptotic theory for high R_0 . If they are, the errors associated with the direct-interaction approximation involve only the value of $f(0)$, the quantitative behaviour of $f(k/k_d)$, and the quantitative functional dependence of $r(k, \tau)$, $g(k, \tau)$ on their argument $v_0 k \tau$. Unfortunately, little can be said at present on the question of corrections for experimental deviations, and we shall attempt only a very qualitative discussion.

We shall not discuss here the deviation of laboratory homogeneous turbulence from maximal randomness, except to note that for grid-produced turbulence we may expect to observe downstream regularities of some kind associated with the grid spacing. We shall, however, consider briefly the relation between our stationary isotropic turbulence and freely decaying isotropic turbulence, which possibly can be closely approximated in the laboratory. For R_0 sufficiently high that the asymptotic solution developed in §7 has a range of accurate validity, it seems clear that the behaviour of this range should be negligibly affected by decay, provided enough time has elapsed for the high wave-number structure to have been fully established. This follows from the fact that the decay of v_0 is very slow compared to characteristic periods of the range. The spectrum and time correlation for the stationary case, therefore, should apply to the free decay case, provided v_0 and ϵ are measured within a time short compared to the decay time.

Since the instantaneous rate of viscous dissipation equals the instantaneous rate of decrease of kinetic energy, it should not matter whether ϵ is measured as $-\partial(\frac{3}{2}v_0^2)/\partial t$ or as $\int_0^\infty 2\nu k^2 E(k) dk$, if accurately isotropic and homogeneous conditions have been established.

At lower Reynolds numbers the equivalence of the stationary and decaying cases no longer follows, and we must consider also that (7.1) no longer is necessarily an accurate expression of the stationary theory. After the numerical constants are evaluated, (6.5) may be of value in the qualitative understanding of lower Reynolds numbers.

The role of anisotropy in the present theory is quite interesting. Let us consider huge Reynolds number homogeneous axisymmetric turbulence. For this case the response and time-correlation functions for high wave-vectors can be determined in close analogy to the isotropic case, yielding similar Bessel function expressions in the direct-interaction approximation. It is easy to see, however, that the effective mixing velocity, replacing v_0 in the arguments, depends on the angle between \mathbf{k} and the axis of symmetry and is maximum when \mathbf{k} is in the direction of maximum r.m.s. velocity. The energy transport among a triad \mathbf{k} , \mathbf{p} , \mathbf{q} in the inertial or dissipation ranges has a consequent dependence on the orientations of these wave-vectors relative to the axis of symmetry. Thus, despite the localness of energy cascade, it does not appear likely that the present theory yields asymptotic isotropy of energy partition in the inertial and dissipation ranges. The geometry of the transport terms is sufficiently complicated that further investigation is needed to determine whether the anisotropy in the θ factors actually results in appreciable energy anisotropy at high wave-numbers and, if so, whether it is of the same or opposite sign as the anisotropy of the r.m.s. velocity components.

As we go to wave-numbers $k \gtrsim R_0 k_0$, the transport terms begin to be dominated by isotropic factors of the form $\exp(-\nu k^2 s)$ in the response functions. In consequence of the localness of the cascade process, we should then expect an approach to energy isotropy at these very high wave-numbers regardless of possible anisotropy lower in the spectrum.

The presence of inhomogeneity as well as anisotropy brings into question the basic assumptions of the present theory. If, for example, we consider turbulence confined within a region not large compared to k_0^{-1} , the mode density will not be high, and we cannot expect the weak dependence property to hold accurately. †

Without further investigation, extreme caution must be used in attempting to interpret shear flow experiments on the basis of homogeneous theory. Laboratory 'spectrum' measurements involve a kind of 'local Fourier analysis' and give results which depend on the position in the inhomogeneous flow. The Fourier

† However, the present theory may be adapted to give a valid description of turbulence homogeneous in only one or two directions—for example, flow in an infinitely long pipe or channel. To obtain closed, determinate covariance equations, weak dependence need be assumed only for wave-vectors differing in their axial components; the phase correlations in transverse directions may be retained and determined, along with the mean velocity profile.

components used in the present theory are *not* local quantities; they refer to the entire flow. Without further justification, we cannot divide into local regions an inhomogeneous flow, in which there are necessarily phase correlations between pairs even of high-lying Fourier modes, and attempt to treat each region more or less independently. It is conceivable that the mixing action of a strong shear velocity in one part of the flow may influence the energy balance in high wave-number Fourier components which affect a laboratory 'spectrum' measurement made in a different part of the flow.

Even apart from the interpretation of the present theory, the concept of local Fourier analysis is not a simple one. Terms like 'eddies of wave-number k ', sometimes used in discussing turbulence, imply a vagueness of distinction between x -space and k -space concepts which can lead to serious inconsistencies.

We shall now proceed to disregard the cautions just presented and compare the asymptotic maximally random, stationary, isotropic theory with real experiments including shear flow experiments. It is hoped that the comparison may lead to insights regarding the questions which have been raised. A further motivation is that the only other course open at present would appear to be to ignore entirely the relation of theory to experiment.

8.2. Wave-number spectra in turbulent pipe flow

The highest laboratory values of R_0 ($> 10^3$) appear to have been obtained in shear flows. An experimental situation for which very careful measurements have been taken is the fully developed flow in a long circular pipe (Laufer 1954). Measurements were made by Laufer of the one-dimensional spectra $\phi_1(k_1)$, $\phi_2(k_1)$, $\phi_3(k_1)$ of (twice) the energy in the axial, radial, and circumferential velocity components, respectively, at several stations across the pipe. The results of the measurements on the axis are shown in figure 5. The one-dimensional spectrum of the total kinetic energy $\phi(k_1) = \phi_1(k_1) + \phi_2(k_1) + \phi_3(k_1)$, for two stations, is shown in figure 6. In most cases the points shown on the latter plot are not experimental points but are computed from the curves drawn by Laufer through the original experimental points for the three component-spectra. In both figures 5 and 6, wire-length corrections have been applied only to $\phi_1(k_1)$. The values of $R_0^{\frac{1}{2}}$, k_0 and k_d shown on figure 6 are computed from $v_0 = 3^{-\frac{1}{2}} \langle \tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2 \rangle^{\frac{1}{2}}$ and estimates of local values of ϵ . The dissipation rate cannot be determined precisely from the measurements, but the computed parameters are not very sensitive to its value. $R_0^{\frac{1}{2}}$ and k_d probably are not in error by more than 10% and k_0 probably by not more than 25%. The accuracy of the relative values of the component spectra may be estimated as about 15 to 20% (Laufer, private communication).

It can be seen from figure 6 that the total energy spectrum follows a $k_1^{-\frac{3}{2}}$ law quite closely for a substantial fraction of the range between k_0 and k_d . If estimated wire-length corrections (Laufer, private communication) are applied to $\phi_2(k_1)$ and $\phi_3(k_1)$, the range of adherence to the $k_1^{-\frac{3}{2}}$ law is somewhat extended. Within the range where $\phi(k_1)$ exhibits ($k_1^{-\frac{3}{2}}$)-law behaviour, it can be seen from figure 5 that there are substantial differences among the slopes of the component spectra, the deficiency in excitation of u_2 and u_3 at low wave-numbers changing to an excess, above the isotropic relation to u_1 , at high wave-numbers.

Superficially at least, the behaviour would seem not inconsistent with what might be expected on the basis of the asymptotic isotropic results. The total energy cascade appears to proceed according to the isotropic law. As the cascade goes on, there appears to be a drain of energy from u_1 to u_2 and u_3 so that the component spectra have respectively greater and lesser slopes than the total

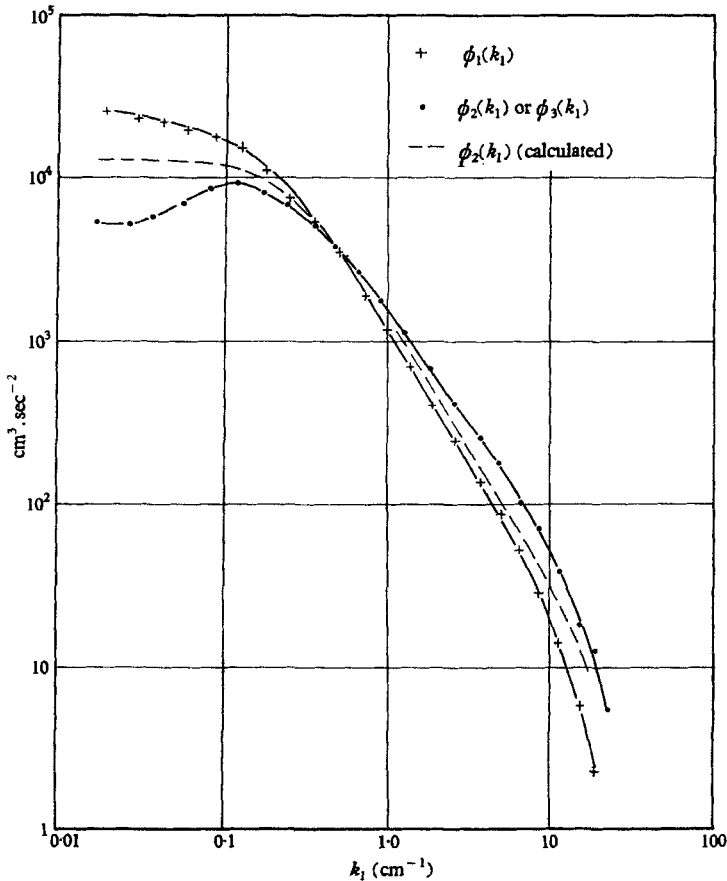


FIGURE 5. Energy-component spectra $\phi_1(k_1)$, $\phi_2(k_1)$, $\phi_3(k_1)$ on the axis of fully turbulent pipe flow. The dashed line gives $\phi_2(k_1)$ as calculated from $\phi_1(k_1)$ using isotropic relations. The pipe radius is $a = 12.33$ cm, and the Reynolds number based on a and the mean velocity on axis is 5×10^5 . After Laufer (1954).

spectrum. The eventual non-isotropic partition of energy at high wave-numbers does not appear too surprising in view of the discussion in §8.1. We may note that $R_0 k_0$, the wave-number at which the response functions must begin to become isotropic, is far above the range plotted.

In his interpretation of the measurements Laufer noted that $\phi_1(k_1)$ follows a $k_1^{-1/2}$ law very closely over a substantial range. This is the inertial range law predicted by the Kolmogorov theory for isotropic turbulence, and the result would seem to indicate consistency with that theory. Actually, the agreement is not very satisfying, as Laufer himself observed. In the centre of the inertial range, $\phi_1(k_1)$ represents only about 20 to 25 % of the total energy associated with k_1 , and

the components $\phi_2(k_1)$, $\phi_3(k_1)$, which contain the rest of the energy, do not follow the $(-\frac{5}{3})$ -law at all. In making a comparison with an isotropic theory it would appear more reasonable to consider the total energy spectrum, as we have done above. For this spectrum, the $(-\frac{3}{2})$ -law appears to give an optimum straight-line fit, and the $(-\frac{5}{3})$ -law appears to lie slightly outside the experimental error. It may also be noted that the very fact of the observed anisotropy at very high wavenumbers, which is not surprising on the basis of the present theory, seems a complete mystery on the Kolmogorov theory.

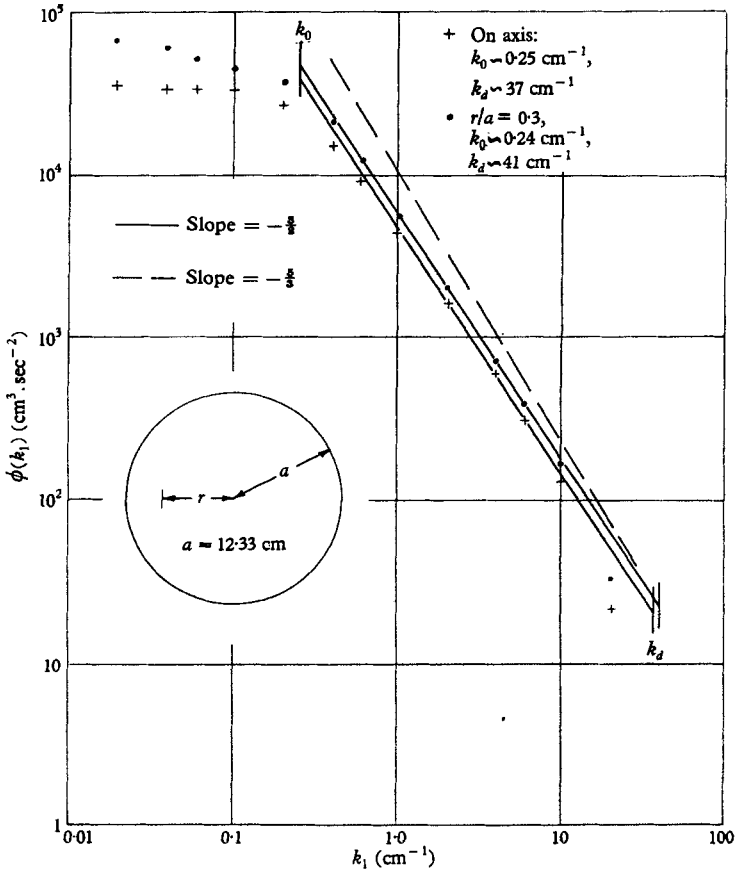


FIGURE 6. Total energy spectrum $\phi(k_1) = \phi_1(k_1) + \phi_2(k_1) + \phi_3(k_1)$ at two stations in the flow of figure 5. $R_0^{\frac{1}{2}} \sim 15$, on axis, ~ 17 at $r/a = 0.3$. From data of Laufer (1954).

It should be emphasized that the discussion above is merely suggestive. The cautions expressed in §8.1 must be kept firmly in mind in evaluating the arguments we have just advanced.

Measurements in the boundary layer on a flat plate at values of $R_0^{\frac{1}{2}}$ similar to those in the pipe-flow experiment have been reported by Klebanoff (1955). In the outer part of the boundary layer a behaviour qualitatively similar to that described above was found; the slope of $\phi_1(k_1)$ was steeper, and that of $\phi_2(k_1)$ flatter, than $-\frac{3}{2}$. Unfortunately, measurements of $\phi_3(k_1)$ were not made. It is

of interest that the anisotropy at very high wave-numbers found in the pipe flow measurements did not appear in this experiment.

Betchov (1957) has also reported measurements at closely comparable $R_0^{\frac{1}{2}}$, made in a shear flow of novel character. Only $\phi_1(k_1)$ was measured, and it resembled closely the corresponding function for the pipe flow. No detailed comparison with isotropic theory seems possible in the absence of data on $\phi_2(k_1)$ and $\phi_3(k_1)$.

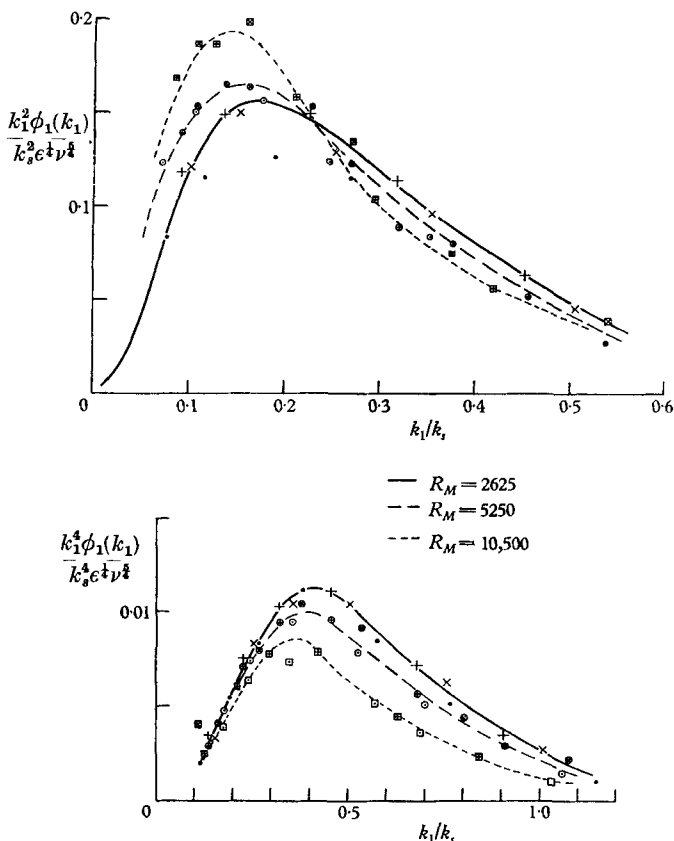


FIGURE 7. Dimensionless plots of $k_1^2 \phi_1(k_1)$ and $k_1^4 \phi_1(k_1)$ according to Kolmogorov scaling ($k_s = R_0^{\frac{1}{2}} k_0 = \epsilon^{\frac{1}{2}} \nu^{-\frac{1}{2}}$), for several grid Reynolds numbers R_M . R_M is defined as (mean stream velocity) \times (grid spacing)/ ν . After Stewart & Townsend (1951).

8.3. Wave-number spectra in grid turbulence

Grid-produced turbulence appears to represent at present the closest laboratory approach to homogeneity and isotropy. Although Reynolds numbers high enough to produce an inertial range have not been described in the literature, it is of some interest to examine to what extent the dissipation range spectrum at moderate Reynolds numbers obeys the scaling indicated by (7.4*b*). Figure 7 shows the results of one-dimensional spectrum measurements by Stewart & Townsend (1951) as plotted and drawn by these authors on the basis of scaling indicated by the Kolmogorov theory. Similar measurements have been reported by Liepmann, Laufer & Liepmann (1951). Figure 8 shows the curves drawn by

Stewart & Townsend through their data points, rescaled according to (7.4*b*). The plots have been prepared so that the curves for $R_M = 5250$ are congruent in each case. The values of R_M correspond to the values of relevant parameters shown in the table below:

R_M	R_0	$R_0^{\frac{1}{2}} = R_0 k_0/k_d$	$R_0^{-\frac{1}{2}} = k_0/k_d$
2625	13	2.35	0.18
5250	26	2.95	0.11
10500	53	3.75	0.07

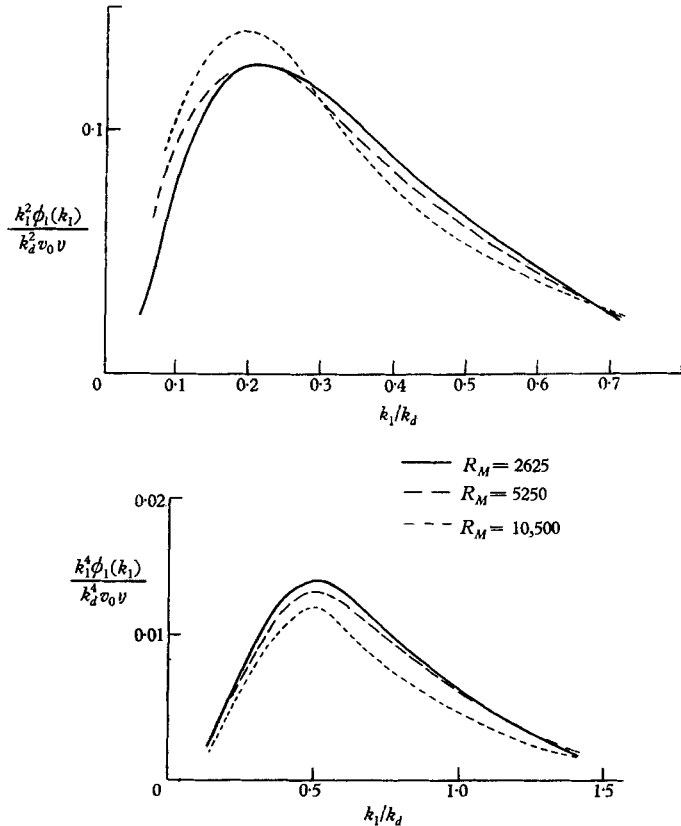


FIGURE 8. The curves in figure 7 rescaled according to the present theory. The rescaling is defined by $k_s = R_0^{\frac{1}{2}} k_d$, $\epsilon^{\frac{1}{2}} \nu^{\frac{1}{2}} = R_0^{-\frac{1}{2}} v_0 \nu$.

It will be noticed that the curves of both $k_1^2 \phi_1(k_1)$ and $k_1^4 \phi_1(k_1)$ fall substantially closer together with the new scaling than with the old. In particular, the systematic shift with R_M of the horizontal position of the maxima seems largely absent with the new scaling. It should be noted that, while $R_0 k_0$ has a definite theoretical significance as the wave-number at which $\nu k^2 = v_0 k$, k_0 is only nominally the wave-number characterizing the energy-containing range. In this experiment most of the energy lies below k_0 in each case. From the values given in the table it would seem not implausible that, particularly in the case of $k_1^4 \phi_1(k_1)$, the functions should bear some relation to the asymptotic forms for a substantial part of the k -range plotted. However, the values of $R_0^{\frac{1}{2}}$ are too low, and the experimental

scatter too high, to permit attaching very much significance to the comparison. It must also be noted that detailed measurements on the isotropy of the flow were not presented.

8.4. *The response and time-correlation functions*

Direct experimental measurements on the modal impulse-response and time-correlation functions $g(k, \tau)$ and $r(k, \tau)$ do not appear to have been made. Indirect information about $r(k, \tau)$ is obtainable, in principle, from measurements of acoustic and electro-magnetic scattering from turbulence, but existing data of this kind appear to involve too many unknowns to provide useful estimates. However, it appears possible to re-express $g(k, \tau)$ and $r(k, \tau)$ so that fairly good indirect estimates of both these quantities for $k_0 \ll k \ll R_0 k_0$ can be made from measurements of the one-point, one-time velocity distribution. Work on this is still incomplete, but it seems pertinent to include a brief report of the tentative conclusions in the present paper. Our discussion here will be confined to the isotropic case.

It seems almost certain on intuitive grounds that for $k_0 \ll k \ll R_0 k_0$ both $g(k, \tau)$ and $r(k, \tau)$ are determined by the energy-range mixing action. That is to say, they are determined by the non-linear terms in (2.2) for which \mathbf{k}' or \mathbf{k}'' falls in the energy-containing range. Now if our double inequality holds, the characteristic fluctuation and relaxation times for energy-range modes will be extremely long compared to those for wave-numbers the order of k . Therefore, if we retain in (2.2) only the terms cited, we have equations of motion for $\mathbf{u}(\mathbf{k})$ and the amplitudes in the neighbourhood of \mathbf{k} which are linear in the amplitudes in this neighbourhood and in which the energy-range amplitudes appear as effectively constant coefficients. These equations then may be handled by the general methods for linear systems with constant coefficients.

The analysis suggested by the last paragraph leads without much difficulty to a simple exact asymptotic expression for $g(k, \tau)$. The result is

$$g(k, \tau) = M(k\tau) \quad (\tau \geq 0, \quad k_0 \ll k \ll R_0 k_0), \quad (8.1)$$

where M is the characteristic function associated with the one-point, one-time distribution function of any velocity component \tilde{u}_1 of the turbulence. M is defined by

$$M(\alpha) = \int_{-\infty}^{\infty} \exp(i\alpha\tilde{u}_1) P(\tilde{u}_1) d\tilde{u}_1 = \langle \exp(i\alpha\tilde{u}_1) \rangle, \quad (8.2)$$

where $P(\tilde{u}_1)$ is this distribution function (Batchelor 1953, §2.3).

The evaluation of $r(k, \tau)$ by these methods is not so direct. It appears to be necessary to appeal explicitly to the maximal randomness condition, which need not be invoked to obtain (8.1). The tentative asymptotic result which we have obtained is

$$r(k, \tau) = M(k\tau) \quad (k_0 \ll k \ll R_0 k_0), \quad (8.3)$$

which might have been anticipated from (8.1). Neither (8.1) nor (8.3) involve the direct-interaction approximation.

It follows from (8.3) that the frequency spectrum function for wave-number k is given by

$$\mathfrak{r}(k, \omega) = 2k^{-1}P(\omega/k) \quad (k_0 \ll k \ll R_0 k_0). \quad (8.4)$$

This result has a very simple interpretation: the weight with which any frequency ω appears in the spectrum function is proportional to the probability of occurrence of the velocity at which a harmonic component of wave-number k would have to be convected to give oscillations of this frequency at a fixed point in space. This interpretation must not be taken too literally, however. The energy-range mixing is not a simple convection process.

Measurements indicate that $P(\tilde{u}_1)$ is close to Gaussian in a variety of turbulent flows (Batchelor 1953, §8.1). For a Gaussian distribution,

$$P(\tilde{u}_1) = (2\pi)^{-\frac{1}{2}} v_0^{-1} \exp(-\frac{1}{2} \tilde{u}_1^2 / v_0^2)$$

and

$$M(\alpha) = \exp(-\frac{1}{2} v_0^2 \alpha^2).$$

For this case (8.3) and (8.4) yield

$$r(k, \tau) = \exp(-\frac{1}{2} v_0^2 k^2 \tau^2), \quad (8.5)$$

$$\mathfrak{r}(k, \omega) = (2/\pi)^{\frac{1}{2}} (v_0 k)^{-1} \exp(-\frac{1}{2} \omega^2 / v_0^2 k^2), \quad (8.6)$$

for the range of k considered. These functions are compared in figures 2 and 3 with the results we have previously obtained by the direct-interaction approximation. It seems likely that the deviations which appear are large compared to those associated with departure of the relevant empirical distribution function from exact Gaussian form and give a fairly good picture of the error due to the direct-interaction approximation. The comparisons suggest that the oscillations in $r(k, \tau)$ and the sharp cut-off in $\mathfrak{r}(k, \omega)$ found in §5 are to be ascribed to the direct-interaction approximation rather than the physical phenomenon.

It is of interest that replacing the integrand factors in (7.2) with the corresponding Gaussian functions (8.5) induces only a small change in $\theta_I(k, p, q)$. For either $p = q = k$ or $p = k, q = 0$ the substitution increases $\theta_I(k, p, q)$ by 5% and for $p = q = \frac{1}{2}k$, by 10%. This might seem to suggest that the direct-interaction approximation gives rather accurate results for $f(k/k_d)$. However, in order to decide this, the error in the functional form (4.3) for $S(k|p, q)$ must also be estimated. We shall discuss this in a future paper.

9. Comparison with other theories

9.1. The Kolmogorov theory

The Kolmogorov theory† has occupied a central place in thinking on turbulence in the past decade because of the intuitive appeal of its assumptions, the economy of its methods, and the approximate empirical support for its predictions. Certain results of the present theory appear to differ only slightly from those of the Kolmogorov theory: the spectrum law in the inertial range by $(k/k_0)^{\frac{1}{2}}$ and the characteristic scaling wave-number in the dissipation range by $R_0^{-\frac{1}{2}}$. However, we shall see that the two theories are in fundamental conflict. First, we shall point out that the observed structure of turbulence at high Reynolds numbers does not seem to support very well the fundamental physical assumptions of the Kolmogorov theory, as distinct from the conclusions of the theory.

† A review of this theory is given by Batchelor 1953, Chapter 6.

Kolmogorov's basic assumption (Kolmogorov 1941) is essentially that the internal dynamics of the sufficiently fine-scale structure (in x -space) at high Reynolds numbers should be independent of the large-scale motion. The latter should, in effect, merely convect, bodily, regions small compared to the macroscale. According to the restatement of this hypothesis by v. Weizsäcker (1948), the dynamical interaction of the $\mathbf{u}(\mathbf{k})$ is effectively local in wave-number space. The direct action of low- k modes on sufficiently high- k modes is assumed to be only a trivial convection effect which does not influence high- k energy dynamics. This implies that the transport of energy from the energy-containing range to the dissipation range depends on a cascade process whose mechanism is independent of the energy-containing range. An intimately related further assumption is that the cascade acts like a diffusion process in k -space. This has the consequence that, whatever the situation at low k , the \mathbf{k} -spectrum should be isotropic and universal in form at sufficiently high k . The equivalent assumption in the x -space picture is that the velocity structure within any region sufficiently small compared to the macroscale should be isotropic and universal in a reference frame moving at the mean velocity of the region ('local isotropy'). If the assumptions are all valid, the dynamical equilibrium at wave-numbers well above the energy-containing range must be determined solely by two local parameters: the rate of cascade ϵ and the viscosity ν . In the inertial range, where one assumes $E(k)$ is independent of ν , simple dimensional arguments then yield the Kolmogorov law $E(k) \propto \epsilon^{2/3} k^{-5/3}$.

Let us suppose that the fine-structure of turbulence consisted of little patches of velocity corrugation, well separated from each other, superimposed on a macrostructure which varied slowly and smoothly with position. Then it would be very hard to see physically how the Kolmogorov assumption about the action of large-scale structure on small-scale could be incorrect. In the limit of the macroscale very large compared to the patch size the action would of necessity be a simple convection. The empirical evidence, however, is quite opposite to this picture. A prominent feature of turbulence at high Reynolds numbers is the presence of sharply defined, extended, and tangled vortex sheets and filaments (Batchelor 1953, §8.4). The high-wave-number Fourier amplitudes are in substantial measure associated with these structures, and the transfer of energy appears to involve, in part, the stretching and thinning of the sheets and filaments by the flow. This behaviour is easily observed by stirring ink into a bathtub of water ($R_0 > 10^4$ is readily achieved). A typical filament extends throughout a substantial part of the turbulent domain. Thus the fine structure does not appear to be 'local' in a three-dimensional sense, and the stretching actions indicate that the fine-scale dynamics may actually depend on the structure of the velocity field over rather large regions.

The vortex sheets may be thought of as internal 'boundary layers' across which occur sharp and major changes in velocity. If these structures form a prominent feature of the turbulence it seems difficult to attach a definite physical meaning to the Kolmogorov notion of a co-ordinate system moving with the local mean velocity. It is hard to know what to take as the velocity of this frame in the regions of rapid change, which, we have noted, contribute importantly to the high- k spectrum. The concept of local isotropy also does not seem well supported

by this physical picture. The vortex sheets are intrinsically anisotropic and inhomogeneous structures. It is only when averages are taken over substantial volumes of the flow, large enough to include many sheets in all orientations, that homogeneity and isotropy can appear, if they do at all. It is very important to note here that *isotropy of space-correlations, for separations small compared to the macroscale, does not imply isotropy within typical regions of the flow which are small compared to the macroscale.* The correlation functions, even for very small separations, are defined by averages over large space regions, or over equivalent ensembles (the former average being the one related to experimental procedure).

Now let us interpret the Kolmogorov theory in the light of the weak dependence principle. Let \mathbf{k} , \mathbf{p} , \mathbf{q} be three distinct modes, all high above the energy-containing range, such that $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$. The energy exchange among these modes is determined by quantities of the type $Q = u_i(\mathbf{k}) u_j(\mathbf{p}) u_m(\mathbf{q})$. Let us examine how Q changes with time. By the weak dependence principle

$$\begin{aligned} \langle Q^* Q' \rangle &= \langle u_i^*(\mathbf{k}) u_j^*(\mathbf{p}) u_m^*(\mathbf{q}) u'_i(\mathbf{k}) u'_j(\mathbf{p}) u'_m(\mathbf{q}) \rangle \\ &= \langle u_i^*(\mathbf{k}) u'_i(\mathbf{k}) \rangle \langle u_j^*(\mathbf{p}) u'_j(\mathbf{p}) \rangle \langle u_m^*(\mathbf{q}) u'_m(\mathbf{q}) \rangle \quad (\text{indices not summed}). \end{aligned} \quad (9.1)$$

It follows that the time characterizing the variation of Q must be of the order of one of the individual characteristic times for the amplitudes $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(\mathbf{p})$, $\mathbf{u}(\mathbf{q})$. The latter times would seem necessarily to be of order $(v_0 k)^{-1}$, $(v_0 p)^{-1}$, $(v_0 q)^{-1}$, respectively, corresponding to the sweeping of structures of the given wavenumbers by the macro-motion. Thus the weak dependence principle leads immediately to the conclusion that the time-variation of the rate of energy-transfer among high-lying modes is strongly dependent on the parameter v_0 , which measures the excitation in the energy-containing range.

The Kolmogorov theory implies that the fluctuations in Q induced by the energy-containing range, although strong, do not affect appreciably the value of $\langle Q \rangle$, which is related to the *mean* energy-transfer among modes \mathbf{k} , \mathbf{p} , and \mathbf{q} . If so, the fluctuations simply reflect the dynamically insignificant distortion of small-scale structures by the slight shear associated with the macro-motion. That they arise at all is then an indication of the essential awkwardness of representing, in Eulerian k -space, phenomena which are more naturally described in quasi-Lagrangian co-ordinates.

In the direct-interaction approximation, on the other hand, these fluctuations represent the dominant contribution to a *relaxation* process acting on the phase correlation among modes \mathbf{k} , \mathbf{p} , and \mathbf{q} built up by their direct coupling. Thus, they depress the magnitude of $\langle Q \rangle$ and inhibit mean energy-transfer. This difference underlies the divergence between the predictions of the two theories. As an illustration, we may replace v_0 in expression (7.2) by $[kE(k)]^{\frac{1}{2}}$, which is a measure of the r.m.s. velocity associated with wave-numbers of order k only. This has the effect of eliminating the influence of the energy-containing range on the relaxation time $\theta_I(k, p, q)$. With the change, it may be verified that (7.1) leads to the inertial-range law and dissipation-range scaling of the Kolmogorov theory.

It is hardly surprising that the direct-interaction approximation does not reproduce the Kolmogorov theory. The convection-without-appreciable-distortion appealed to by that theory involves the 'co-operative' action of the

very many triad interactions which link the energy-containing modes with pairs of the high-wave-number Fourier components of the convected small-scale structures. It is likely to be described poorly in an approximation where the various triad interactions are related only through relaxation effects. More surprising is the small difference between the spectrum laws of the two theories. Whatever may turn out to be the exact inertial range law, this in itself perhaps justifies a degree of confidence in the essential soundness of the direct-interaction approximation. It implies that even in the situation assumed by Kolmogorov, where the convection effects which the direct-interaction approximation cannot represent are very strong, their neglect does not much affect the ultimate energy equilibrium. From this point of view our application to the high-wave-number range at large R_0 constitutes a rather severe trial of the approximation. One might expect improved results at wave-numbers below the inertial range, where convection-without-distortion clearly does not occur in any event.

At the present time it seems difficult to decide between the two sets of inertial and dissipation range predictions. In fact, the exact inertial range law may be neither $k^{-\frac{3}{2}}$ nor $k^{-\frac{5}{3}}$. We have pointed out earlier some unrealistic aspects of the physical picture usually associated with the Kolmogorov theory. Perhaps a more serious drawback to this approach is the difficulty of developing it to the point of yielding quantitative predictions beyond the existing dimensional results. On the other hand, if the influence of v_0 on the inertial-range mean-energy-dynamics turns out to be an artifact of the direct-interaction approximation, the absolute inertial and dissipation range spectrum levels obtainable by this approximation may be considerably in error at sufficiently high R_0 . It should be possible to shed some light on these questions by examining the next approximation in the sequence described briefly at the end of § 2.4.

9.2. Heisenberg's heuristic theory

Several authors have proposed models for the transport power $\Pi(k)$ based on simple physical analogies (Batchelor 1953, Chapter 6). Of these the heuristic theory of Heisenberg (1948) has aroused the most interest. Heisenberg assumed that $\Pi(k)$ could be represented as a power dissipation by all wave-numbers smaller than k due to the action of an effective 'eddy viscosity' which represented their coupling to all wave-numbers larger than k . In analogy to the expression for dissipation by real viscosity, he took

$$\Pi(k) = \int_0^k 2 \left\{ \kappa \int_k^\infty [q^{-3} E(q)]^{\frac{1}{2}} dq \right\} p^2 E(p) dp, \quad (9.2)$$

where κ is a disposable numerical constant. The eddy viscosity $\{ \}$ was obtained by dimensional arguments on the basis of the Kolmogorov-v. Weizsäcker hypothesis of the localness of k -space dynamics. Because of its dimensional structure, (9.2) leads to a $k^{-\frac{3}{2}}$ -inertial range spectrum law. At very high k it leads to a k^{-7} -law, implying the non-existence of mean-square third- and higher-order velocity derivatives.

The comparison of the eddy viscosity and the present theories is made more

illuminating by first modifying (9.2) to yield a $k^{-\frac{3}{2}}$ inertial range law; Heisenberg's basic physical assumption seems independent of the conflict between the direct-interaction approximation and the Kolmogorov theory. Accordingly, we shall consider the transport function

$$\Pi(k) = \int_0^k 2 \left\{ \kappa' v_0^{-1} \int_k^\infty q^{-1} E(q) dq \right\} p^2 E(p) dp, \quad (9.3)$$

where κ' is a new disposable numerical constant. The eddy viscosity $\{ \}$ is here obtained by dimensional considerations under the requirement that it depend inversely on v_0 . The algebraic rationality of (9.3) makes it perhaps more satisfying intuitively than (9.2).

From (9.3) we immediately obtain

$$-\frac{\partial \Pi(k)}{\partial k} = 2\kappa' k(v_0 k)^{-1} \left[\int_0^k \frac{p^3}{k} E(k) E(p) \frac{dp}{p} - \int_k^\infty k^2 E(k) E(p) \frac{dp}{p} \right]. \quad (9.4)$$

This equation has been written in a form to facilitate comparison with $-\partial \Pi(k)/\partial k$ as given by the right side of (7.1). The time factor $(v_0 k)^{-1}$ corresponds to the θ_I factor in (7.1). The most immediate qualitative difference between the two expressions is that (9.4) expresses the total interaction as an integral over *pair* interactions (k, p) , rather than the *triad* interactions (k, p, q) which enter (7.1), and which are fundamental to the structure of the original equation of motion (3.1). A further difference concerns the sharp distinction in (9.4) between the interactions with modes higher than or lower than k . The absorption term involves only $p < k$ and the emission term only $p > k$. In (7.1) both sorts of modes enter both terms, and the net input appears as the balance of both ingoing and outgoing contributions from every triad interaction; this more closely resembles other dissipative equilibrium situations in statistical physics.

$E(k)$ appears as a factor in both the absorption and emission terms of (9.4), suggesting that this transport expression does not give the tendency to return to equilibrium, upon a perturbation of the spectrum in the neighbourhood of k , that was noted in the discussion of (7.1). This peculiarity shows up more clearly in (9.3) and also in the unmodified form (9.2). If the equilibrium is disturbed by an initial increase of spectrum level of wave-numbers above k , then by (9.2) or (9.3) the effective eddy viscosity acting on wave-numbers below k is increased. This will lead to an increased flow of energy from wave-numbers below k to those above k which will tend to increase further the eddy viscosity acting on the lower wave-numbers. A kind of instability seems to be indicated. On general statistical mechanical grounds we should expect precisely contrary behaviour; an increase of excitation in modes above k should lead to a decreased flow of energy to them from modes below k , as found in §4.

The comparisons made above can also be made on the basis of (9.2) but with less clarity. For this case the algebraic irrationality of the eddy viscosity expression results in greater asymmetry between absorption and emission terms.

9.3. *The analytical theories of Heisenberg and Chandrasekhar*

Heisenberg (1948, §5) and Chandrasekhar (1955) have developed analytical theories of turbulence based on the approximation that fourth-order moments

of the two-time distribution of the $\mathbf{u}(\mathbf{k})$ are related to second-order moments as if the $\mathbf{u}(\mathbf{k})$ were independent or, equivalently, that the corresponding x -space moments are related as in a normal distribution. We shall call this the quasi-normality approximation. It is far more drastic than the direct-interaction approximation, for the cross-moments involved. Consequently, we may surmise immediately, on the basis of the discussion in §9.1, that these theories should not yield the Kolmogorov theory. This seems to be supported by the attempt of Chandrasekhar (1956) to reconcile his theory with the $k^{-\frac{5}{3}}$ inertial range law. In order to obtain the formal possibility of a space correlation function consistent with this law, Chandrasekhar found it necessary to assume that the full space-time correlation function for small space-time separations was independent of v_0 in ordinary Eulerian co-ordinates. The Kolmogorov theory requires, instead, that this be true in a quasi-Lagrangian system moving with the macro-motion (Kolmogorov 1941), and therefore not true in Eulerian co-ordinates. As we shall note below a related situation arises in Heisenberg's theory.

Heisenberg's theory properly is appropriate to freely decaying turbulence, but its essential structure may be discussed in terms of our present specialization to the stationary case. The two cases should be equivalent, in any event, for treatment of the inertial and dissipation ranges at high Reynolds numbers. The theory first involves expressing the triple moment $S(k, \tau)$, defined by (3.5), in terms of fourth-order moments by substituting for $u_m(\mathbf{q}, t)$ a bilinear integral expression obtained by integrating (2.2) (with $\mathbf{k} \rightarrow \mathbf{q}$, $i \rightarrow m$) with respect to time. Next, the fourth-order moments are expressed in terms of $U(k, \tau)$ by means of the quasi-normality approximation. This results in a closed equation for $U(k, \tau)$. The analysis was carried out for $\nu = 0$, but can be straightforwardly generalized to $\nu \neq 0$.

The expression which results for $S(k, \tau)$ can be obtained from our result (3.7) by (a) replacing the response functions $g(p, s)$, $g(q, s)$ with $\exp(-\nu p^2 s)$, $\exp(-\nu q^2 s)$; (b) discarding the terms involving $a(k, p, q)$; (c) retaining from the terms involving $b(k, p, q)$ only a particular linear combination of the parts which are antisymmetric in k and p for $\tau = 0$. The alteration of the response functions is equivalent to ignoring the effect on them of the non-linear interaction. The discarding of terms amounts, on the basis of the present theory, to throwing out most of the interaction altogether. This shows up strikingly if one evaluates the contribution to $r(k, \tau)$, for high k , due to energy-range mixing, after the fashion in which (5.5) and (5.6) were obtained. One finds that the dependence of $r(k, \tau)$ on v_0 disappears as $k/k_0 \rightarrow \infty$. This result apparently led Heisenberg to the following conjecture: Independence of $r(k, \tau)$ on v_0 is physically unreasonable in Eulerian co-ordinates, and moreover conflicts with the Kolmogorov theory. However, it is just the behaviour one would expect, on the Kolmogorov theory, in a quasi-Lagrangian system moving with the low- k motion. Therefore, possibly the quasi-normality approximation, and the resulting equation of motion (Heisenberg 1948, equation (83)), may be appropriate to such a co-ordinate system. The difficulty with this conjecture would seem to be that, aside from whether or not the quasi-normality approximation is appropriate to a Kolmogorov-type co-ordinate system, it is hard to see how making the approximation could automatically effect a transformation to such co-ordinates.

The theory of Chandrasekhar (1955) was developed by analysis in x -space. Expressed in k -space, it differs from that of Heisenberg, as generalized above, only in that $S(k, \tau)$ is expressed in terms of fourth-order moments by substituting a bi-linear expression for $u_i^*(\mathbf{k}, t - \tau)$ rather than for $u_m(\mathbf{q}, t)$. This gives, however, quite a different result from Heisenberg's, thereby indicating the intrinsic inconsistency of the quasi-normality approximation. Chandrasekhar's result may be obtained from (3.7) by replacing $g(k, s)$ with $\exp(-\nu k^2 s)$ and discarding all the contributions involving $b(k, p, q)$. The terms retained are all rigorously non-negative for $\tau = 0$, and, in contrast to Heisenberg's prescription, the theory consequently does not yield conservation of energy by the Reynolds stresses and pressure forces (Kraichnan 1957).

Our brief discussion has indicated that in both Heisenberg's and Chandrasekhar's theories the use of the quasi-normality approximation is equivalent to discarding essential parts of the dynamical coupling of the Fourier modes. The unphysical result for $r(k, \tau)$ on Heisenberg's theory and the major violation of energy conservation on Chandrasekhar's seem, in some respects, almost as serious as the unphysical time-dependence and violation of energy conservation implied by neglect of the third-order moments at the outset. We conjecture that this points towards a basic property of the dynamical system and that significant improvement should not be expected from theories made determinate by ignoring higher-order cumulants instead of fourth-order. As the order increases, the number of individual cross-moments of the $\mathbf{u}(\mathbf{k})$ entering the dynamically relevant expressions rises rapidly, and their total importance cannot be expected to decrease with increase of order, except for a very short time after a statistically independent initial state of the $\mathbf{u}(\mathbf{k})$, or at very low Reynolds numbers. This situation is connected with the non-linearity of (2.2) which results in phase relations among large groups of modes being established by networks of triad interactions. The direct-interaction approximation, on the other hand, is not equivalent to neglect of cumulants of any finite order. It is an approximation on the dynamics, rather than directly on the statistical distribution resulting from the dynamics.

9.4. *The theories of Proudman & Reid and Tatsumi*

Proudman & Reid (1954) and Tatsumi (1957) make the quasi-normality approximation only for the distribution of simultaneous values of the $\mathbf{u}(\mathbf{k}, t)$. They are able, thereby, to obtain closed equations for the time change of $E(k)$ in freely decaying isotropic turbulence. No predictions are obtained concerning time correlations. The analytical structure of these authors' theories is complicated, and we shall only indicate very briefly the nature of the relation with the present theory for the inertial range at very high Reynolds numbers. It should be noted that these theories were primarily intended to describe the energy-containing range.

We shall assume that there exists an inertial range, far above the energy-containing range, in which energy transport takes place by local cascade and in which the direct effects of viscosity are negligible. In this range we shall assume that the spectrum obeys a k^{-n} -law. Then it is not difficult to verify that the

approximation of Proudman & Reid and Tatsumi can be used to obtain an asymptotic expression for $-\partial\Pi(k)/\partial k$ at time t which differs from the right side of (7.1) only in that $\theta_I(k, p, q)$ is replaced by

$$\tau_0 = \int_t^\infty \frac{E(k, s)}{E(k, t)} ds, \quad (9.5)$$

where $E(k, t)$ is the value of $E(k)$ at time t .

From empirical knowledge we know that τ_0 must be of order $(v_0 k_0)^{-1}$, the overall decay time of the turbulence, whereas we have found that θ_I is of order $(v_0 k)^{-1}$ for significant triad interactions. If arguments are followed similar to those which led to and verified (6.3), we now find instead

$$E(k) \sim (ev_0 k_0)^{\frac{1}{2}} k^{-2} \quad (9.6)$$

for the asymptotic inertial range spectrum.† With this spectrum, however, $2\nu k^2 E(k)$ is independent of k . This is not inconsistent with the concept of an inertial range (which requires $1 < n < 3$), but it does seem an unlikely asymptotic property according to present experimental indications.

The results suggested above are not too surprising. The quasi-normality approximation for the one-time distribution is known empirically to be considerably in error for high wave-numbers (Batchelor 1953, §8.2). The appearance of τ_0 instead of θ_I in the transport expression would seem to indicate that, as in the case of the two-time quasi-normality approximation, the effect of ignoring fourth-order cumulants is to discard most of the effect of the dynamical interaction on the time-correlation functions and response functions. The theories of Proudman & Reid and Tatsumi do yield exact conservation of energy by the non-linear interaction.

The author would like to thank the many persons whose valuable comments and criticism contributed to this paper. Particular thanks are due to Dr G. K. Batchelor, Dr W. V. R. Malkus, Dr I. Proudman and Dr R. A. Silverman for discussions of the foundations of the theory. Mr R. Wernick and Dr M. Yanowitch have contributed to the mathematical analysis. Dr J. Laufer has kindly provided information on experimental errors and wire-length corrections associated with the measurements discussed in §8.2. Several authors have kindly given permission for the reproduction of figures from their published papers. Finally, the author would like to express his indebtedness to the authors whose theories are analysed in §9. The concepts developed in these theories played an essential role in forming the ideas underlying the present paper.

The research reported herein received support from the Geophysics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, United States Air Force under contract AF 19(604)2138.

† Dr Tatsumi has recently informed the author that he also has obtained a spectrum which varies as k^{-2} , working from the equations of motion of his theory.

REFERENCES

- BATCHELOR, G. K. 1953 *The Theory of Homogeneous Turbulence*. Cambridge University Press.
- BETCHOV, R. 1957 *J. Fluid Mech.* **3**, 205.
- CHANDRASEKHAR, S. 1955 *Proc. Roy. Soc. A*, **229**, 1.
- CHANDRASEKHAR, S. 1956 *Phys. Rev.* **102**, 941.
- HEISENBERG, W. 1948 *Z. Physik*, **124**, 628.
- VON KÁRMÁN, T. 1948 *C.R. Acad. Sci., Paris*, **226**, 2108.
- KLEBANOFF, P. S. 1955 *Nat. Adv. Comm. Aero., Wash.*, Rep. 1247.
- KOLMOGOROV, A. N. 1941 *C.R. Acad. Sci. U.S.S.R.* **30**, 301; **32**, 16.
- KOVÁSZNAY, L. S. G. 1948 *J. Aero. Sci.* **15**, 745.
- KRAICHNAN, R. H. 1957 *Phys. Rev.* **107**, 1485.
- KRAICHNAN, R. H. 1958a *Phys. Rev.* **109**, 1407. [Cited as Paper I.] Errata: **111**, 1747.
- KRAICHNAN, R. H. 1958b *Physics of Fluids*, **1**, 358.
- LAUFER, J. 1954 *Nat. Adv. Comm. Aero., Wash.*, Rep. 1174.
- LIEPMANN, H. W., LAUFER, J. & LIEPMANN, K. 1951 *Nat. Adv. Comm. Aero., Wash.*, Tech. Note 2473.
- MILLIONSHTCHIKOV, M. 1941 *C.R. Acad. Sci. U.S.S.R.* **32**, 615, 619.
- OBUKHOV, A. M. 1941 *C.R. Acad. Sci. U.S.S.R.* **32**, 19.
- PROUDMAN, I. & REID, W. H. 1954 *Phil. Trans. A*, **247**, 163.
- STEWART, R. W. & TOWNSEND, A. A. 1951 *Phil. Trans. A*, **243**, 359.
- TATSUMI, T. 1957 *Proc. Roy. Soc. A*, **239**, 16.
- V. WEIZSÄCKER, C. F. 1948 *Z. Physik*, **124**, 614.